Exercises for TATA55, batch 2, 2021

October 21, 2021

Solutions to the exercises below should be handed in no later than November 16, 2021.

1. (6p) Let $\phi : G \to H$ be a group homomorphism, with kernel N and image M.

   (a) Show that the group homomorphism

   $$\begin{aligned} C_\infty = \langle g \rangle &\to D_n = \langle r, s | r^n = s^2 = rsrs = 1 \rangle \\ g^k &\mapsto r^k \end{aligned}$$

   can be "factored" through the quotient epimorphism

   $$\begin{aligned} C_\infty = \langle g \rangle = \{ g^m | m \in \mathbb{Z} \} &\to C_n = \langle h \rangle = \left\{ h^k \middle| 0 \leq k < n \right\} \\ g^k &\mapsto h^k \end{aligned}$$

   (b) More generally, show that ϕ can be factored as $\phi = \hat{\phi} \circ \pi$, where $\pi : G \to G/N$ is the canonical quotient epimorphism, and $\hat{\phi}$ is injective.

   (c) Show that the homomorphism of additive groups

   $$\begin{aligned} \mathbb{Z} &\to \mathbb{Z} \\ n &\mapsto 2n \end{aligned}$$

   can be "factored" through the inclusion homomorphism $2\mathbb{Z} \hookrightarrow \mathbb{Z}$.

   (d) Show that ϕ factors as $\phi = j \circ \tilde{\phi}$, where j is the inclusion homomorphism, and $\tilde{\phi}$ is surjective.

   (e) Show that the group homomorphism $\mathbb{R} \ni t \mapsto \exp(2\pi i t) \in \mathbb{C}^*$ can be factored as

   $$\begin{aligned} \mathbb{R} \twoheadrightarrow \frac{\mathbb{R}}{\mathbb{Z}} \qquad\qquad &\to \mathcal{T} \hookrightarrow \mathbb{C}^* \\ t \mapsto t + \mathbb{Z} \mapsto \exp(2\pi i t) &\mapsto \exp(2\pi i t) \end{aligned}$$

   where the first homomorphism is surjective, and the last is injective.

   (f) Can ϕ be factored as $j \circ \bar{\phi} \circ \pi$, with j injective, π surjective, and $\bar{\phi}$ an isomorphism?

2. (4p)

   (a) Show that the group of complex invertible 2×2 matrices acts on the stereographic one-point compactification of the complex plane by

   $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} . z = \frac{az + b}{cz + d}$$

(b) What are the orbits?

(c) What are the fixed points of a general (generic) matrix?

(d) What is the stabilizer of i?

3. (3p) Suppose that $K \leq H \leq G$ are finite groups, where K, H are not necessarily normal subgroups of G. Show that
$$|G : K| = |G : H| \cdot |H : K|$$

4. (4p) Let G be the group of rigid symmetries of the cube. Pick two adjacent faces of the cube, let r_1 be rotation by 90 degrees ccw through an axis perpendicular to the face (so r_1 has order 4) and let r_2 be the corresponding rotation of the adjacent face. Let $H = \langle r_1 r_2, r_2 r_1 \rangle$. What is the size of H? What well-known group is it isomorphic to? Is it normal in G? If so, what is the quotient?

5. (4p) Let G be a finite group of size pm, with p prime, and all prime factors of m larger than p. Suppose that $H \leq G$, $|G : H| = p$. Show that H is normal in G.

   (Hint: G acts on the left cosets of H. This action is equivalent to a homomorphism $G \to S_Y$ for some Y. Show that the kernel is contained in H.)

6. (6p) Consider two $m \times n$-matrices A and B as equivalent if by permuting the rows and columns of A we can obtain B. Suppose that the matrices are binary, i.e. have entries in $\{0, 1\}$.

   - Count the number of equivalent binary 2×3-matrices with k ones, for $0 \leq k \leq 6$.
   - Count the number of equivalent binary 3×4-matrices with k ones, for $0 \leq k \leq 12$.

   The first part is easily doable with pen and paper and Burnside. For the second part, you might want to look up the "cycle index" and "Polya enumeration theorem", and/or start up your computer!