Exercises for TATA55, batch 3, 2021

November 30, 2021

Solutions to the exercises below should be handed in no later than December xxx, 2021.

1. (3p) Let R be a commutative, unitary ring. Let

$$\mathrm{Nil}(R) = \{r \in R | \exists n \geq 1, \, r^n = 0\}.$$

   (a) Show that $\mathrm{Nil}(R)$ is an ideal of R.
   (b) Show that $\mathrm{Nil}(R)$ is not necessarily an ideal of a non-commutative ring R.
   (c) Show that if $r \in \mathrm{Nil}(R)$ then $1 - r$ is invertible in R.

2. (3p) Find the characteristic of the following commutative rings:
   (a) $\frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{9\mathbb{Z}} \times \frac{\mathbb{Z}}{15\mathbb{Z}}$
   (b) $\mathbb{Z}[i]$, where $i \in \mathbb{C}$, $i^2 = -1$
   (c) $\frac{\mathbb{Z}[j]}{(2-5j)}$ where j is a primitive 3rd root of unity, $j^3 = 1$ but, $j^2 \neq 1$, you can explicitly take $j = \exp(\frac{2}{3}\pi i) \in \mathbb{C}$.

3. (2p) Provide explicit ring isomorphisms between
   (a) $\frac{\mathbb{Z}[x]}{(n,x)}$ and $\frac{\mathbb{Z}}{n\mathbb{Z}}$,
   (b) $\frac{\mathbb{Z}[x]}{(n)}$ and $(\frac{\mathbb{Z}}{n\mathbb{Z}})[x]$.

4. (3p) Which of the following ideals in $\mathbb{Z}[x]$ are prime? Which are maximal?
   (a) $(x, x+1)$,
   (b) $(5, x^2+4)$,
   (c) $(x^2+1, x+2)$.

For the two remaining questions, some parts may require the use of a computer. For instance, SAGEmath will make light work of them! Feel free to ask me about SAGEmath!

You can of course use another computer algebra software, or write your own programs.

5. (4p) Let $g(x) = x^6 - x^3 - 2 \in \mathbb{Q}[x]$. Put $R = \mathbb{Q}[x]/(g(x))$.

   (a) Is R an integral domain?

   (b) Find all proper, non-trivial ideals of R.

   (c) Let a denote the coset $x + (g(x)) \in R$. Find, if possible, the inverse of a.

   (d) Find a general expression for a^k, $k \geq 0$, as a linear combination of $a^0, a^1, a^2, a^3, a^4, a^5$.

6. (5p) Let $R = \mathbb{Q}[D_4]$, the group algebra on $D_4 = \langle r, s | r^4 = s^2 = rsrs = 1 \rangle$. In other words, R is the $\mathbb{Q}$-vector space with basis elements labeled with the elements of D_4, and with multiplication the $\mathbb{Q}$-linear extension of the multiplication on basis elements given by the multiplication of D_4.

   (a) Put $t = 1 * r + 1 * s \in R$. Calculate $t * t$ and $t * t * t$

   (b) Put $v = 1 * 1 + 1 * s$. Find an explicit expression for v^k for any positive k.

   (c) Show that the map

$$F : \mathbb{Q}[D_4] \to \mathbb{Q}$$
$$\sum_{g \in D_4} c(g) g \mapsto \sum_{g \in D_4} c(g)$$

   is $\mathbb{Q}$-linear and calculate its kernel.

   (d) Show that the *left annihilator*

$$\mathrm{Ann}(t) = \{ f \in R | f * t = 0 \}$$

   is a left ideal of R, and calculate a basis of it as a $\mathbb{Q}$-vector space.

   (e) List the conjugacy classes in D_4. Calculate the *center* of R, i.e.,

$$\mathrm{Center}(R) = \{ f \in R | f * h = h * f \text{ for all } h \in R \}$$

   Compare.