Exercises for TATA55, batch 4, 2021

December 14, 2021

All answers should be accompanied by a thorough motivation!

1. (3p) Let $\alpha = \sqrt{2} + \sqrt[3]{5}$. Find the minimal polynomial of α over $\mathbb{Q}$ and the degree of the extension $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

2. (4p) Let F be a field with $q < \infty$ elements, and let K be an extension of F.
   (a) Prove that $a^q = a$ for all $a \in F$.
   (b) If $b \in K$ is algebraic over F, show that $b^{(q^m)} = b$ for some $m > 0$.

3. (4p) Let $\alpha \in \mathbb{C}$. Then α is an algebraic integer iff it is the root of an equation of the form
$$\alpha^m + b_1\alpha^{m-1} + \cdots + b_m = 0, \qquad b_1, \ldots, b_m \in \mathbb{Z}$$
   (a) Show that an algebraic integer is algebraic over $\mathbb{Q}$.
   (b) Show that the converse does not hold.
   (c) Show that any element of $\mathbb{C}$ which is algebraic over $\mathbb{Q}$ can be scaled by a positive integer to become an algebraic integer.
   (d) Show that $\sqrt{1/3} + \sqrt[3]{1/5}$ is not an algebraic integer; scale it with a positive integer so that it becomes one.

4. (4p) Recall that a field isomorphism is a ring isomorphism preserving the multiplicative identity, and that a field automorphism is a field isomorphism from the field to itself.
   (a) Prove that complex conjugation is a field automorphism.
   (b) What are the field automorphisms of $\mathbb{Q}$?
   (c) What are the field automorphisms of $\mathbb{Q}(\sqrt[3]{2})$?
   (d) What are the field automorphisms of a field with 27 elements?

5. (4p) Let $\alpha \in \mathbb{C}$, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 5$. Put $\beta = \alpha^3$.
   (a) What is $[\mathbb{Q}(\beta) : \mathbb{Q}]$?
   (b) If $\alpha^5 = \alpha - 1$, what is the minimal polynomial of β?

For the last question the use of a computer is adviced. Ask me about SAGEmath if you want to use that program!

6. (6p) Let $F = \mathrm{GF}(9)$, expressed as $\mathbb{Z}_3[y]/(y^2 + 2y + 2) \simeq \mathbb{Z}_3(a)$.

(a) There are of course 9 irreducible monic linear polynomials in $F[x]$; how many irreducible quadratic polynomials are there?

(b) The following sequence of elements in F is periodic; enough of it is given that you will be able to deduce the period.

$$
\begin{gathered}
(c_j)_{j=0}^{\infty} = (2*a+1, 1, 2, 2*a+2, 2, 2*a, 0, a+1, a+1, 2*a, 2, a, 2*a, \\
a+2, 2*a, 2*a+2, 0, 2*a+1, 2*a+1, 2*a+2, 2*a, a+1, 2*a+2, 1, 2*a+2, a+2, \\
0, 2, 2, a+2, 2*a+2, 2*a+1, a+2, a, a+2, 1, 0, 2*a, 2*a, \\
1, a+2, 2, 1, a+1, 1, a, 0, 2*a+2, 2*a+2, a, 1, 2*a, a, 2*a+1, \\
a, a+1, 0, a+2, a+2, a+1, a, 2*a+2, a+1, 2, a+1, 2*a+1, 0, 1, 1, 2*a+1, \\
a+1, a+2, 2*a+1, 2*a, 2*a+1, 2, 0, a, a, 2, 2*a+1, 1, 2, 2*a+2, 2, 2*a, \\
0, a+1, a+1, 2*a, 2, a, 2*a, a+2, 2*a, 2*a+2, 0, 2*a+1, 2*a+1, 2*a+2, 2*a, a+1, 2*a+2, \\
1, 2*a+2, a+2, 0, 2, 2, a+2, 2*a+2, 2*a+1, a+2, a, a+2, 1, 0, 2*a, 2*a, 1, \ldots)
\end{gathered}
$$

Find this period (and preperiod, if applicable).

(c) Find the recurrence relation over F that this sequence satisfies.

(d) Find the generating function of the sequence.

(e) Factor the denominator of the generating function (over some explicit extension of F), then perform partial fraction decomposition of the generating function.

(f) Find an explicit formula for c_j of the form

$$c_j = u\alpha^j + v\beta^j$$

where u, v, α, β lies in some (explicit) extension of F.

7. (4p) Do the following **instead of** the previous exercise if you found it to hard:

(a) Find an irreducible monic polynomial of degree 3 over $\mathbb{Z}_3$

(b) Use this polynomial to generate a recurrent sequence in $\mathbb{Z}_3$ of period $3^3 - 1$.

(c) Pretend that you are given this mysterious sequence, form the generating function, recognize it as a rational function $f(x)/g(x)$.

(d) Find an explicit formula for the n'th element of the sequence.