Exercises for TATA55, batch 3, 2022

December 6, 2022

Solutions to the exercises below should be handed in no later than Decmber 13, 2022.

We denote by D_n the dihedral group of symmetries of the regular n-gon, so $|D_n| = 2n$.

1 Part one: computer assistance is helpful

In particular, the laboration that is on the course homepage should be easy to modify to perform the necessary calculations.

Even if you prefer to write the exercises by hand, you might want to print out, or better e-mail a text file of, the computer runs.

1. (4p) For a positive integer n, we put $[n] = \{1, 2, \ldots, n\}$, and let $\binom{[n]}{2}$ denote the set of ordered pairs in $[n]$.

   A simple graph on n is determined by its edge set $E \subseteq \binom{[n]}{2}$. Two graphs E_1, E_2 are isomorphic if there is a permutation $\phi \in S_n$ such that

   $$E_2 = \{ (\phi(u), \phi(v)) | (u, v) \in E_1 \}$$

   For $n = 2, 3, 4, 5, 6$ count the number of non-isomorphic graphs. Also count the sizes of the various isomorphism classes, and give representatives; draw pictorial presentations of these representatives at least for $n \leq 4$.

2. (4p) We can regard a simple graph on $[n]$ with edge set $E \subseteq \binom{[n]}{2}$ as follows: all potential edges are present, those in E are colored black, and are visible, those in $\binom{[n]}{2} \setminus E$ are colored white, and are invisible. An immediate generalisation is: what if we color the potential edges using k colors, where k can be larger than 2? Use Burnsides lemma and/or Polya-theory to find the number of non-equivalent k-colored complete graphs on n vertices, for $n = 2, 3, 4, 5, 6$, and arbitrary $k \geq 2$. If you use the code on the course homepage, briefly explain what it does.

3. (6p) By picking 3 vertices on a regular n-gon we define an inscribed triangle; there are $\binom{n}{3}$ such triangles. The dihedral symmetries of the n-gon induces an action on the set of those triangles. Find the number and sizes of these orbits, for $4 \leq n \leq 15$. Guess (prove it if you can) the number and sizes of these orbits for general n.

2 Part two: no computer necessary

For this part, you may check your results using a computer, but you should do the exercises by hand. You may refer to any theorem and result in your texbook(s), prove your other assertions.

4. (3p) List the conjugacy classes in A_4.

5. (4p) How many permutations in S_n, $n \geq 4$, commute with $(1, 2, 3, 4)$?

6. (4p) Find the Sylow subgroups of D_6.

7. (4p) Show that if p is an odd prime, then any Sylow p-subgroup of D_n is cyclic and normal.

8. (3p) An automorphism on a group G is a group isomorphism from G to G. Find the number of automorphism on C_n, the cyclic group of order n.

9. (5p) Find the number of automorphisms on $C_2 \times C_2$. Then find the number of automorphisms on $C_3 \times C_3$ and finally on $C_p \times C_p$ where p is an odd prime.

10. (4p) Let G be a finite group of size pm, with p prime, and all prime factors of m larger than p. Suppose that $H \leq G$, $|G : H| = p$. Show that H is normal in G. (Hint: G acts on the left cosets of H. This action is equivalent to a homomorphism $G \to S_Y$ for some Y. What is the kernel?)

11. (4p) Let $R = \mathbb{C}[x, y]$. A monomial ideal $I \subset R$ is an ideal that can be generated by monomials $x^a y^b$. Show that a monomial ideal has the following property:

$$\sum_{a=0}^{m} \sum_{b=0}^{n} c_{ab} x^a y^b \in I \qquad \Longleftrightarrow x^a y^b \in I \text{ for all } a, b \text{ such that } c_{a,b} \neq 0$$

    Conversely, show that such an ideal is a monomial ideal.

12. (4p) Show that any monomial ideal in R is the ideal theoretic sum of finitely many principal monomial ideals.

13. (4p) What are the primary ideals in R? Write (x^2y^3, xy^5) as an intersection of primary ideals.