Exercises for TATA55, batch 1, 2023

September 20, 2023

1. (4p) Find all solutions to

$$2x + 3y + 5z = 2001, \qquad x, y, z \in \mathbb{Z}$$

   Which solutions have x, y, and z positive?

   Solution: The integers 2,3,5 are pairwise coprime, and a particular solution is $X_p = 2001(-1, 1, 0)$. The general homogeneous solution is $s(-3, 2, 0) + t(-5, 0, 2)$ so the general solutin is

$$(x, y, z) = (-2001, 2001, 0) + s(3, -2, 0) + t(-5, 0, 2), \qquad s, t \in \mathbb{Z}.$$

   For positivity we get

$$-2001 + 3s - 5t > 0, \quad 2001 - 2s > 0, \quad 2t > 0$$

   which simplifies to

$$0 < t < \frac{2001}{10}, \quad \frac{2001}{3} + \frac{5}{3}t < s < \frac{2001}{2}$$

2. (4p) Solve

$$\begin{aligned} x &\equiv 2 \mod 13 \\ x &\equiv 3 \mod 17 \\ x &\equiv 4 \mod 19 \end{aligned}$$

   Solution: Since the moduli are distinct primes, there is a unique solution mod $13 * 17 * 19 = 4199$. We can find the solution by solving a linear diophantine equation in 3 variables; we get that $x \equiv 4032 \mod 4199$.

3. (4p) Show the following induction principle: suppose that $S \subseteq \mathbb{N} \times \mathbb{N}$ satisfies

   (a) $(0, 0) \in S$

   (b) If $(a, b) \in S$ then $(a + 1, b) \in S$ and $(a, b + 1) \in S$

Then $S = \mathbb{N} \times \mathbb{N}$.

Solution: Put

$$S_m = \{ (a, b) \in S | a + b = m \}$$
$$T_m = \{ (a, b) \in \mathbb{N} \times \mathbb{N} | a + b = m \}$$

We want to prove that $S_m = T_m$ for all $m \in \mathbb{N}$.

Clearly, $S_0 = T_0$. Suppose that $S_n = T_n$. If $(c, d) \in \S_{n+1}$ then either $c > 0$ or $d > 0$. If $c > 0$ then since $(c - 1, d) \in S_n$ it follows that $(c, d) \in S_{n+1}$, and similarly if $d > 0$. Hence $S_{n+1} = T_{n+1}$. The result follows by (ordinary) induction.

4. (4p) Suppose that $\gcd(a, n) = d$, and that $d\,|b$. Show that the congruence $ax \equiv b \mod n$ is equivalent to the congruence $(a/d)x \equiv b/d \mod n/d$. How many different solutions (mod n) does this congruence have? Find all non-congruent solutions to the congruence $10x \equiv 35 \mod 25$.

   Solution: Write $a = da', b = db', n = dn'$. Then

   $$n\,|ax - b \iff n'\,\big|a'x - b'$$

   which happens iff $a'x \equiv b' \mod n'$ is solvable. Since $\gcd(a', n') = 1$ this congruence has a unique solution mod n'; there are d congruence classes mod n which are congruent to this x mod n.

   The congruence $10x \equiv 35 \mod 25$ is equivalent to $2x \equiv 7 \mod 5$, which has the unique (mod 5) solution $x \equiv 1 mod 5$. Mod 25 the solutions are $x \equiv 1, 6, 11, 16, 21 \mod 25$.