

# Exercises for TATA55, batch 2, 2023

September 12, 2023

Solutions to the exercises below should be handed in no later than September 26, 2023.

1. (4p) How many subgroups of size  $k$  are there in  $C_n$ ?
2. (6p) Same question for the dihedral group  $D_n$  (partial credit for partial results).
3. (4p) Let  $G$  be a group, and suppose that  $a^2 = 1$  for all  $a \in G$ . Show that  $G$  is abelian. On the other hand, show that the relations  $a^3 = b^3 = 1$  for a group  $G$  generated by  $a, b$  does not imply that the group is abelian.
4. (5p) Denote the adjacent transposition  $(j, j + 1)$  by  $s_j$ .
  - (a) Show that the set  $\{s_1, \dots, s_{n-1}\}$  generate  $S_n$ .
  - (b) Find the relations (including self-relations) among these generators.
  - (c) Show that the set of all  $t_{ij} = s_i s_j$  generate  $A_n$ .
  - (d) Show that the set of all  $u_j = (12j)$  generate  $A_n$ .
  - (e) Find the relations between the  $u_j$ 's.
5. (3p, a bit harder) Let  $\sigma \in S_n$  be a permutation of cycle type  $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_r]$ . Let  $V = \mathbb{C}^n$  with canonical basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  and denote by  $T_\sigma : \mathbb{C}^n \rightarrow \mathbb{C}^n$  the linear map that satisfies  $T_\sigma(\mathbf{e}_j) = \mathbf{e}_{\sigma(j)}$ . What are the eigenvalues of  $T_\sigma$ ? Start with the case where  $\sigma$  is  $k$ -cycle.