Exercises for TATA55, batch 3, 2023

October 12, 2023

Solutions to the exercises below should be handed in no later than November xxx, 2023.

1. (3p) If N and M are normal subgroups of G show that also
$$NM = \{nm | n \in N,\ m \in M\}$$
is a normal subgroup of G.

2. (3p) Let N be a normal subgroup of the finite group G, and let $a \in G$. Show that the order $o(aN)$ of the coset $aN \in G/N$ divides the order $o(a)$ of a in G.

3. (3p) It is an important theorem that for $n > 4$, the normal subgroups of S_n are the trivial group 1, S_n, and A_n, with corresponding quotients S_n, 1, and C_2. But what about $n = 4$?

4. (3p) What are the possible orders $o(f)$ for a bijection $f : \mathbb{Z} \to \mathbb{Z}$, regarded as an element of $S_{\mathbb{Z}}$? What if we know, in addition, that f fixes a countably subset of $\mathbb{Z}$, the complement of which is also countable?

5. (3p) The punctured complex plane $C^* = \mathbb{C} \setminus \{0\}$ is a group under multiplication. Show that any non-trivial disc
$$B(z_0, r) = \{z \in \mathbb{C}^* | |z - z_0| < r\}$$
generates $\mathbb{C}^*$ as a group, i.e. that $\langle B(z_0, r) \rangle = \mathbb{C}^*$.

6. (3p) Show that subgroups of $\mathbb{C}^*$ invariant under all rotations around the origin correspond bijectively to subgroups of the group of positive real numbers under multiplication.