Solutions to exercises for TATA55, batch 3, 2023

November 5, 2023

1. (3p) If N and M are normal subgroups of G show that also
$$NM = \{ nm \ : \ n \in N, \ m \in M \}$$
is a normal subgroup of G.

Solution: First, note that $1 \in NM$. Secondly we show that $NM = MN$. Take $n \in N, m \in M$. Since M is normal, $mn = nn^{-1}mn = nm_2 \in NM$ with $m_2 \in M$. Hence $MN \subseteq NM$, and the reverse inclusion follows similarly.

Thirdly, if $n_1, n_2 \in N$, $m_1, m_2 \in M$ then
$$(n_1 m_1)(n_2 m_2) = n_1 (m_1 n_2) m_2 = n_1 (n_3 m_3) m_2 \in NM$$
where we used that $MN = NM$.

Fourthly, if $n \in N, m \in M$ then $(nm)^{-1} = m^{-1}n^{-1} \in MN = NM$ since M, N are closed under inverses.

Finally, if $n \in N, m \in M, g \in G$ then
$$gnmg^{-1} = gng^{-1}gmg^{-1} = (gng^{-1})(gmg^{-1}) = n_2 m_2 \in NM$$
sine N, M are normal subgroups.

2. (3p) Let N be a normal subgroup of the finite group G, and let $a \in G$. Show that the order $o(aN)$ of the coset $aN \in G/N$ divides the order $o(a)$ of a in G.

Solution: Let $n = o(a)$. Then $a^n = 1$, so $(aN)^n = a^n N = 1N = N$. For any group it holds that if $x^n = 1$ then $o(x) \,|n$. Hence the order of aN divides n.

3. (3p) It is an important theorem that for $n > 4$, the normal subgroups of S_n are the trivial group 1, S_n, and A_n, with corresponding quotients S_n, 1, and C_2. But what about $n = 4$?

Solution: The conjugacy classes of S_4 are
$$\{()\}, \qquad \{(12), (13), (23), (14), (24), (34)\},$$
$$\{(12)(34), (13)(24), (14)(23)\}, \qquad \{(123), (124), (134), (234)\},$$
$$\{(1234), (1243), (1324), (1342), (1423), (1432)\}$$
Normal subgroups are union of conjugacy classes, but if we include transpositions, they generate S_4. Same with the fourcycles. The threecycles generata all even permutations, thus give the normal subgroup A_4. However, we can also take
$$N = \{()\} \cup \{(12)(34), (13)(24), (14)(23)\}$$

which is a normal subgroup of index 6. Putting $a = (12)$, $b = (13)$ we observe that the commutator

$$aba^{-1}b^{-1} = abab = (123) \notin N,$$

so the quotient S_4/N is non-abelian. Up to isomorphism, there are but two groups of order 6, namely C_6 and S_3, so $S_4/N \simeq S_3$.

4. (3p) What are the possible orders $o(f)$ for a bijection $f : \mathbb{Z} \to \mathbb{Z}$, regarded as an element of $S_{\mathbb{Z}}$? What if we know, in addition, that f fixes a countably subset of $\mathbb{Z}$, the complement of which is also countable?

   Solution: The bijection

$$a_n(k) = \begin{cases} k & k \leq 0 \text{ or } k > n \\ 2 & k = 1 \\ \vdots & \vdots \\ n & k = n-1 \\ 1 & k = n \end{cases}$$

   has order n. The bijection $n \mapsto n+1$ has infinite order.

   Now for examples with countable fixpointset with countable complement. The bijection

$$b_n(k) = \begin{cases} k & k \text{ odd} \\ k+2 & k \text{ even} \end{cases}$$

   has infinite order, and fixes precisely the odd integers.

   The followng bijection g_n fixes precisely the non-positive integers, and has order n:

$$g_n(k) = \begin{cases} k & k \leq 0 \\ 2 & k = 1 \\ \vdots & \vdots \\ n & k = n-1 \\ 1 & k = n \\ n+2 & k = n+1 \\ \vdots & \vdots \\ 2n & k = 2n-1 \\ n+1 & k = 2n \\ 2n+2 & k = 2n+1 \\ \vdots & \vdots \end{cases}$$

   So, any order, finite or not, can be achieved, even in the restricted scenario.

5. (3p) The punctured complex plane $C^* = \mathbb{C} \setminus \{0\}$ is a group under multiplication. Show that any non-trivial disc

$$B(z_0, r) = \{ z \in \mathbb{C}^* \ : \ |z - z_0| < r \}$$

   generates $\mathbb{C}^*$ as a group, i.e. that $\langle B(z_0, r) \rangle = \mathbb{C}^*$.

Solution: First recall that $\langle S \rangle$ contains all finite products of elements of S and their inverses, as well as the identity. So

$$\frac{1}{z_o} B(z_0, r) = B(1, r) \subseteq \langle B(z_0, r) \rangle.$$

Next, take any positive $s \in \mathbb{R}$ such that $e^{is} \in B(1, r)$. Then there is a $\delta > 0$ and a positive integer N such that

$$\cup_{j=0}^{n} e^{ijs} B(1, r) \supseteq \{ z \in \mathbb{C}^* \ : \ 1 - s < |z| < 1 + s \} =: \text{Annulus}(1 - s, 1 + s)$$

However,

$$\text{Annulus}(1 - s, 1 + s) * \text{Annulus}(1 - s, 1 + s) \supseteq \text{Annulus}((1 - s)^2, (1 + s)^2)$$

et cetera, and these ever increasing annuli cover $\mathbb{C}^*$.

6. (3p) Show that subgroups of $\mathbb{C}^*$ invariant under all rotations around the origin correspond bijectively to subgroups of the group of positive real numbers under multiplication.

   Solution: Consider the surjective map

$$F : \mathbb{C}^* \to \mathbb{R}_{>0}$$
$$F(z) = |z|$$

Since $|zw| = |z||w|$, this is a group homomorphism from $\mathbb{C}^*$ to the group of positive real numbers, under multiplication.

The kernel is the circle group $\mathcal{T} = \{ z \in \mathbb{C}^* \ : \ |z| = 1 \}$. So, by the correspondence theorem, subgroups of $\mathbb{R}_{>0}$ correspond to subgroups of $\mathbb{C}^*$ that contain $\mathcal{T}$.

Now, if a subgroup $G \leq C^*$ is invariant under <u>all</u> rotations, for any $z \in G$ the whole circle $\{ w \in \mathbb{C}^* \ : \ |w| = |z| \}$ must be contained in G. In particular, $\mathcal{T} \subseteq G$.

Conversely, if $\mathcal{T} \subseteq G$, then for any $z \in G$, and any $e^{i\phi} \in \mathcal{T}$, it holds that $e^{i\phi} z \in G$. Thus, G is invariant under rotations.