## Exercises for TATA55, batch 4, 2023

## October 23, 2023

Solutions to the exercises below should be handed in no later than November xxx, 2023. You may use a computer.

1. (4p) Let N = 2n with n a positive integer, and let  $[N] = \{1, 2, ..., N\}$ . The symmetric group  $S_N$  acts on [N] in the natural way. Show that this induces an action on  $\binom{N}{n}$ , subsets of [N] of cardinality n, by

$$\sigma.\{a_1,\ldots,a_n\}=\{\sigma(a_1),\ldots,\sigma(a_n)\},$$

and thus on the set

$$X = \{\{A, B\} | A \cap B = \emptyset, A \cup B = [N], |A| = |B| = n\}$$

Use this to prove that

$$\mathsf{K} = \{ \sigma \in \mathsf{S}_{\mathsf{N}} | \sigma.\mathsf{V} = \mathsf{V} \text{ for all } \mathsf{V} \in \mathsf{X} \}$$

is a normal subgroup of  $S_N$ . Find this subgroup for n = 2, 3 and describe it, and the corresponding quotient  $S_N/K$ .

- 2. (6p) The tiles of a 4x4 chessboard are colored either red or blue. How many non-equivalent colorings are there, under the symmetries induced by
  - (a) Cyclic permutation of columns
  - (b) Simultaneous cyclic permutations of rows and columns
  - (c) dihedral symmetry?
- 3. (3p) A simple graph on a finite set X is determined by its edge set  $E \subseteq {\binom{X}{2}}$ . Two such graphs are isomorphic if there is a permutation  $\sigma \in S_X$  such that

$$E_2 = \sigma.E_1 = \{\{\sigma(a), \sigma(b)\} | \{a, b\} \in E_1\}.$$

How many isomorphism classes of simple graphs are there, if |X| = 4? If |X| = 5?

4. (3p) We can generalize the concept of a simple graph on X be coloring the edges with k colors. Such a k-colored graph can be described by a map  $f : \binom{X}{2} \to [k]$ ; one of the colors is used to indicate that the potential edge is not present in the graph. To such graphs f, g are ismorphic if there is a  $\sigma \in S_X$  such that  $f = g \circ \sigma$ .

How many isomorphisms classes of k-colored graphs are there on two vertices? On three vertices?