## Exercises for TATA55, batch 4, 2023

October 23, 2023

Solutions to the exercises below should be handed in no later than November xxx, 2023. You may use a computer.

1. (4p) Let $N=2 n$ with $n$ a positive integer, and let $[N]=\{1,2, \ldots, N\}$. The symmetric group $S_{N}$ acts on $[N]$ in the natural way. Show that this induces an action on $\binom{N}{n}$, subsets of $[N]$ of cardinality $n$, by

$$
\sigma .\left\{a_{1}, \ldots, a_{n}\right\}=\left\{\sigma\left(a_{1}\right), \ldots, \sigma\left(a_{n}\right)\right\}
$$

and thus on the set

$$
X=\{\{A, B\}|A \cap B=\emptyset, A \cup B=[N],|A|=|B|=n\}
$$

Use this to prove that

$$
\mathrm{K}=\left\{\sigma \in \mathrm{S}_{\mathrm{N}} \mid \sigma . \mathrm{V}=\mathrm{V} \text { for all } \mathrm{V} \in \mathrm{X}\right\}
$$

is a normal subgroup of $S_{N}$. Find this subgroup for $n=2,3$ and describe it, and the corresponding quotient $S_{N} / K$.
2. (6p) The tiles of a $4 x 4$ chessboard are colored either red or blue. How many non-equivalent colorings are there, under the symmetries induced by
(a) Cyclic permutation of columns
(b) Simultaneous cyclic permutations of rows and columns
(c) dihedral symmetry?
3. (3p) A simple graph on a finite set $X$ is determined by its edge set $E \subseteq\binom{X}{2}$. Two such graphs are isomorphic if there is a permutation $\sigma \in S_{X}$ such that

$$
\mathrm{E}_{2}=\sigma \cdot \mathrm{E}_{1}=\left\{\{\sigma(\mathrm{a}), \sigma(\mathrm{b})\} \mid\{\mathrm{a}, \mathrm{~b}\} \in \mathrm{E}_{1}\right\} .
$$

How many isomorphism classes of simple graphs are there, if $|X|=4$ ? If $|X|=5$ ?
4. (3p) We can generalize the concept of a simple graph on $X$ be coloring the edges with $k$ colors. Such a k-colored graph can be described by a map $f:\binom{X}{2} \rightarrow[k]$; one of the colors is used to indicate that the potential edge is not present in the graph. To such graphs $f, g$ are ismorphic if there is a $\sigma \in S_{X}$ such that $f=g \circ \sigma$.
How many isomorphisms classes of k-colored graphs are there on two vertices? On three vertices?

