Solutions for Exercises for TATA55, batch 4, 2023

December 2, 2023

1. (4p) Let $N = 2n$ with n a positive integer, and let $[N] = \{1, 2, \ldots, N\}$. The symmetric group S_N acts on $[N]$ in the natural way. Show that this induces an action on $\binom{N}{n}$, subsets of $[N]$ of cardinality n, by
$$\sigma.\{a_1, \ldots, a_n\} = \{\sigma(a_1), \ldots, \sigma(a_n)\},$$
and thus on the set
$$X = \{\{A, B\} | A \cap B = \emptyset, A \cup B = [N], |A| = |B| = n\}$$
Use this to prove that
$$K = \{\sigma \in S_N \,|\, \sigma.V = V \text{ for all } V \in X\}$$
is a normal subgroup of S_N. Find this subgroup for $n = 2, 3$ and describe it, and the corresponding quotient S_N/K.

Solution: The verifications of the group actions beeing group actions are routine.

For $n = 2$, $N = 4$, we recall the exercise from B3 and that the normal subgroups of S_4 are S_4, A_4, 0, and
$$N = \{()\} \cup \{(12)(34), (13)(24), (14)(23)\}.$$
Since kernels are normal subgroups, the kernel is one of these. We check that $(123) \in A_4$ is not in the kernel, since
$$(123).\{\{1, 2\}, \{3, 4\}\} = \{\{2, 3\}, \{1, 4\}\}$$
On the other hand, $(12)(34)$ is, since
$$(12)(34).\{\{1, 2\}, \{3, 4\}\} = \{\{2, 1\}, \{4, 3\}\} \tag{1}$$
$$(12)(34).\{\{1, 3\}, \{2, 4\}\} = \{\{2, 4\}, \{1, 3\}\} \tag{2}$$
$$(12)(34).\{\{1, 4\}, \{2, 3\}\} = \{\{2, 3\}, \{1, 4\}\} \tag{3}$$
So the kernel K is not trivial, nor does it contain A_4, so it is N. We saw in B3 that $S_4/N \simeq S_3$.

For $n = 3$, $N = 6$, the possible normal subgroups are 0, A_6, S_6. Again we check that $(123) \in A_6$ is not in the kernel, since
$$(123).\{\{1, 2, 4\}, \{3, 5, 6\}\} = \{\{2, 3, 4\}, \{1, 5, 6\}\}$$
Thus the kernel K is trivial.

2. (6p) The tiles of a 4x4 chessboard are colored either red or blue. How many non-equivalent colorings are there, under the symmetries induced by

(a) Cyclic permutation of columns

(b) Simultaneous cyclic permutations of rows and columns

(c) dihedral symmetry?

Solution: In (a), the group is $C_4 = \langle g \rangle$ and $X = 2^{4\times 4}$, so $|X| = 2^{16} = 65536$ If the board is

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

then the action of C_4 gives the following subgroup of S_X:

$$\begin{aligned} g^0 &\to () \\ g^1 &\to (abcd)(efgh)(ijkl)(mnop) \\ g^2 &\to (ac)(bd)(eg)(fh)(ik)(jl)(mo)(np) \\ g^3 &\to (adcb)(ehgf)(ilkj)(mpon) \end{aligned}$$

Polya/Burnside now gives the number of orbits as

$$\frac{1}{4}(2^{16} + 2 * 2^4 + 2^8) = 2^{14} + 2^6 + 2^3 = 16456$$

In (b), we have the same group acting on X, but it is expressed as permutations on the board elements in a different way. Here

$$\begin{aligned} g^0 &\to () \\ g^1 &\to (afkp)(bglm)(chin)(dejo) \\ g^2 &\to (ak)(fp)(bl)(gm)(ci)(hn)(dj)(eo) \\ g^3 &\to (ahkn)(belo)(cfip)(dgjm) \end{aligned}$$

So the number of orbits is once again

$$\frac{1}{4}(2^{16} + 2 * 2^4 + 2^8) = 16456$$

In (c), the dihedral group D_4 induces

$$\begin{aligned} () &\to () \\ r &\to (ampd)(bioh)(cenh)(fjkg) \\ r^2 &\to (ap)(md)(bo)(ih)(cn)(eh)(fk)(jg) \\ r^3 &\to (adpm)(bhoi)(chne)(fgkj) \\ s &\to (m)(j)(g)(d)(ch)(bl)(fk)(ap)(ej)(in) \\ s_2 &\to (bc)(fg)(jk)(no)(ad)(eh)(il)(mp) \\ s_3 &\to \text{type } 1^4 2^6 \\ s_4 &\to \text{type } 2^8 \end{aligned}$$

Hence the number of orbits is now

$$\frac{1}{8}(2^{16} + 2 * 2^4 + 2^8 + 2 * 2^{10} + 2 * 2^8) = 8548$$

3. (3p) A simple graph on a finite set X is determined by its edge set $E \subseteq \binom{X}{2}$. Two such graphs are isomorphic if there is a permutation $\sigma \in S_X$ such that

$$E_2 = \sigma.E_1 = \{\{\sigma(a), \sigma(b)\} | \{a, b\} \in E_1\}.$$

How many isomorphism classes of simple graphs are there, if $|X| = 4$? If $|X| = 5$?

Solution: We add the line

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print(f'n={n}: {len(graphsiso)} graphs\n')
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to the code on the homepage and get 11 graphs on 4 vertices, and 34 on 5 vertices. My super-computer at home was able to calculate that there is a whooping 156 graphs on 6 vertices!

4. (3p) We can generalize the concept of a simple graph on X be coloring the edges with k colors. Such a k-colored graph can be described by a map $f : \binom{X}{2} \to [k]$; one of the colors is used to indicate that the potential edge is not present in the graph. To such graphs f, g are ismorphic if there is a $\sigma \in S_X$ such that $f = g \circ \sigma$.

How many isomorphisms classes of k-colored graphs are there on two vertices? On three vertices?

Solution: : On two vertices there is a single potential edge, which can be given any of the k colors. No need for computer calculations!

On three vertices the code on the homepage calculates the cycle index for S_3 as

$$1/6 * x_0^3 + 1/2 * x_0 * x_1 + 1/3 * x_2,$$

i.e. there is one permutation of cycle type 1^3, 3 of type $1^1 2^1$, and 2 of type 3^1. Polya's theorem tells us that specializing $x_i \to k$ gives us the number of k-colorings, we get

$$1/6 * k^3 + 1/2 * k^2 + 1/3 * k$$

When $k = 2$ this is 4. We go back to the previous program and check that, yes, the number of graphs on 3 vertices is 4.

Now we boldly go one step beyond what was asked of us and plug in $n = 4$. We get

$$1/24 * k^6 + 3/8 * k^4 + 7/12 * k^2$$

colorings, for $k = 2$ this is 11, and we have already calculated that there are 11 graphs on 4 vertices.