Exercises for TATA55, batch 5, 2023

December 1, 2023

Solutions to the exercises below should be handed in no later than December xx, 2023.

1. (3p) Provide an explicit ring isomorphism $\frac{\mathbb{Z}[x]}{(4,6,3x,5x)} \simeq \mathbb{Z}_2$.

2. (3p) Solve the equation (in $\mathbb{Q}[x]$)
$$f(x)(2x^3 + 3x^2 + 7x + 1) + g(x)(5x^4 + x + 1) = x + 3$$

3. (4p) List all ideals in $S = \mathbb{Z}_7[x]/(h(x))$ where $h(x) = x^4 + 2x^2 + 2$. Is S an integral domain?

4. (3p) Factor $11y^5 - 55y^4 + 85y^3 - 30y^2 - 35y + 39 \in \mathbb{Z}[x]$. (Hint: try a linear substitution)

5. (4p) Let $R = \mathbb{Q}[u, v, w]/(u^2v^2 - w^3)$. Find a finitely many monomials $x^{a_j}y^{b_j}$ in $S = \mathbb{Q}[x, y]$ such that $R \simeq \mathbb{Q}[x^{a_1}y^{b_1}, \dots, x^{a_r}y^{b_r}]$.

6. (6p) Show that
   (a) not every function from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2$ is $\mathbb{Z}_2$-linear,
   (b) but all such functions are polynomial,
   (c) and they correspond bijectively to cosets of the ideal
$$(x_1^2 + x_1, x_2^2 + x_2, \dots, x_n^2 + x_n) \subset \mathbb{Z}_2[x_1, \dots, x_n].$$