TATA55 HT2023

Hand-in exam batch 6

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1 Grading, due date

- Hand in your assignment no later than January 20, 2024. I will publish solutions on the course homepage soon after that, so late submissions are inadmissible.
- Ensure that your name is prominently displayed on the first page.
- Each exercise is worth 6 points. Total score exceeding $6 * 4/2 = 12$ is guaranteed a passing grade.
- Take care to provide complete proofs/supporting arguments for all your assertions.
- Best of luck, and merry christmas!

2 Exercises

1. Let $K = \mathbb{Q}(a)$, where a is a root of $a^3 = 2$. Let $L = K(b)$, where b is a root of $b^2 = -3$.
   (a) Find the minimal polynomial of a/b over K, and over $\mathbb{Q}$
   (b) Same for $a + b$
   (c) What is $[L : Q]$?
2. Let $f(x) = x^4 + x^2 + x + 1 \in \mathbb{Z}_3[x]$.
   (a) Show that f is irreducible over $\mathbb{Z}_3$, then factor f over $K = \frac{\mathbb{Z}_3[x]}{(f(x))}$
   (b) Consider the element $a = x + (f(x)) \in K$. What is its (multiplicative) order? Does it generate K^*?
   (c) Find a generator of K^*.

3. Find the splitting fields of the following polynomials. Factor the polynomial in this field, and find the degree of the extension. Prove all your results in excruciating detail! The best way to ensure correctness is to construct a tower of Kronecker extensions.

   (a) $f(x) = x^3 + 2x^2 + 3x + 1 \in \mathbb{Q}[x]$
   (b) $g(x) = x^3 + 2x^2 + 3x + 1 \in \mathbb{Z}_7[x]$
   (c) $h(x) = x^3 + 2x^2 + 3x + 1 \in \mathbb{Z}_{13}[x]$

4. Let n be a positive integer, and let M_n be the set of $n \times n$-matrices with entries in $\mathbb{Z}_3$. Let G_n denote the subset of invertible matrices.

   (a) Calculate M_n and $|G_n|$.
   (b) Calculate the number of matrices in M_n and in G_n with determinant $[2]_3$.
   (c) Calculate the fraction $\frac{|G_n|}{|M_n|}$.