Change of variables

1. $R = k[x]$, k field, $a \in k$.

Then $R \xrightarrow{\varphi} R$
$\varphi(f(x)) = f(x-a)$

is an automorphism

Pf $\varphi(f(x)+g(x)) = \varphi((f+g)(x)) = (f+g)(x-a) = f(x-a)+g(x-a)$
$= \varphi(f(x)) + \varphi(g(x))$

$\varphi(f(x)g(x)) = (fg)(x-a) = f(x-a)g(x-a) = \varphi(f(x))\varphi(g(x))$

$\square$

2 Get iso $\dfrac{k[x]}{(f(x))} \longrightarrow \dfrac{k[x]}{(f(x-a))}$

Pf $k[x] \xrightarrow{\varphi} k[x] \xrightarrow{\pi} \dfrac{k[x]}{(f(x-a))}$

is surjective, kernel $(f(x))$

Ex $\dfrac{\mathbb{Q}[x]}{(x^2-2x+1)} \cong \dfrac{\mathbb{Q}[x]}{(x^2)}$

$\parallel$

$\mathbb{Q}(a)$, $a^2 = 2a-1$ $\qquad$ $\mathbb{Q}(b)$, $b^2 = 0$

$x + ya$ $\qquad$ $u + vb$

Ex

$$C = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \det(C - tI) = t^4 - 12t^3 + 54t^2 - 108t + 81$$
$$= (t-3)^4$$

Cayleigh-Hamilton: $(C - 3I)^4 = 0$

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^4 = 0$$

Actually: $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = 0$, so $(C - 3I)^2 = 0$

<u>Minimal pol</u>: $\quad \mathbb{Q}[x] \xrightarrow{F} \mathrm{Mat}(\mathbb{Q}, 4, 4)$

$$F(x) = F(C)$$

$\operatorname{Im} F$ = all pol. expr. in C, commutative subalgebra

$$\operatorname{Ker} F = ((x-3)^2)$$

$$F(C) \cong \frac{\mathbb{Q}[x]}{((x-3)^2)} \cong \frac{\mathbb{Q}[x]}{x^2}$$

ring of dual nbrs

basis $1, \bar{x}$, $\quad \bar{x}^2 = 0$

Zerodivisors