Abstract Algebra, Lecture 8

Group actions

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Lecture notes availabe at course homepage

http://courses.mai.liu.se/GU/TATA55/

Summary

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1. Definition of group action
2. Examples of group actions
3. Right actions
4. Fixed points, Orbits, Stabilizers
   - Orbits are equivalence classes
   - Stabilizers are subgroups
   - Orbit-stabilizer lemma
5. Burnside's theorem
   - Examples
   - Colorings

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Summary

Remark

I will make lots of accompanying drawings on the whiteboard today, without which these slides will be hard to digest. If you missed the lecture and are reviewing these notes by your lonesome, do attempt to draw some pictures in the marginal as you go!

Definition

Let G be a group and X a set. An action of G on X is a group homomorphism

$$\phi : G \to S_X$$

from G to the group of bijections on X.

Often, one wishes to identify the group element g with the bijection $\phi(g) : X \to X$ that it induces. To that end, we write $g.x$ for $\phi(g)(x)$.

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Lemma

A group action is the same thing as a map

$$G \times X \to X$$
$$(g, x) \mapsto g.x$$

satisfying, for all $g, h \in G$, $x \in X$*:*

1. $1.x = x$,
2. $(gh).x = g.(h.x)$

Proof.

The map defines ϕ by

$$\phi(g)(x) = g.x,$$

since $(g^{-1}g).x = g^{-1}.(g.x)$, $\phi(g$ is a bijection. Clearly, $\phi(1)$ is the identity bijection, and we also see that $\phi(gh) = \phi(g) \circ \phi(g)$. □

Example (Trivial action)

G can act trivially by $g.x = x$ always

Example (Linear maps)

If V is a vector space over $\mathbb{R}$, and G is the group of invertible linear maps on V, then G acts on V by

$$F.\mathtt{v} = F(\mathtt{v})$$

Example (Groups of symmetries)

If $K \subset E^n$ is a convex, centrally symmetric body then
the subgroup G of the linear isometries on E^n that consists of all F such that $F(K) = K$
acts on the points of K.

Example (Dihedral groups)

If $K_n \subset E^2$ is the regular n-gon in standard position, then the dihedral group D_n acts on the points of K_n. By restriction, it also acts on the vertices of K_n, and the action is determined by this restriction.

Example (Shifting)

The infinite cyclic group $C_\infty = \langle g \rangle$ acts on the set of all bi-infinite sequences (of whatever) by

$$g^k.(a_j)_{j \in \mathbb{Z}} = (b_j)_{j \in \mathbb{Z}}, \qquad b_j = a_{j+k}$$

Example (Cyclic shifting)

The cyclic group $C_m = \langle g \rangle$ acts on m-tuples by

$$g.(x_1, \ldots, x_m) = (x_2, x_3, \ldots, x_m, x_1)$$

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Example (Left regular representation)

Any group acts on itself by left multiplication:

$$\phi(g)(x) = g.x = gx$$

This is what we used in Cayley's theorem; note that there it was essential that

$$\phi : G \to S_G$$

was injective. We do not demand this for general group actions; those that satisfy this are called *faithful*.

Example

If $H \leq G$ then H acts on G as above.

Example (Permutation subgroups)

Let X be a finite set, and let $G \leq S_X$. Then G acts on X in the natural way.

Example (Cycle diagram)

If $G = \langle \sigma \rangle$ is a cyclic subgroup of S_n, then the cycle diagram of σ is a graphical depiction of the action of G on X.

Example (Conjugation)

Let G be a group, and let H be a subgroup (it can be G itself). Then H acts on G by conjugation:

$$h.g = hgh^{-1}$$

Example (Cosets)

If $H \leq G$ then G acts on the set of left H-cosets by

$$g.uH = (gu)H$$

Definition

A right action by G on X is a map

$$\begin{aligned} X \times G :&\to X \\ (x, g) &\mapsto x.g \end{aligned}$$

satisfying, for all $g, h \in G$, $x \in X$:

1. $x.1 = x$,
2. $x.(gh) = (x.g).h$

So, the difference to a left action is that when acting with gh, g is applied first, h last.

Definition

If G is a group, then the opposite group G^{op} has the same underlying set as G, and modified multiplication

$$g *_{op} h = hg$$

Definition

Let G, H be groups. An anti-homomorphism $\tau : G \to H$ satisfies $\tau(hg) = \tau(g)\tau(h)$.

Lemma

An anti-homomorphism is nothing but a group homomorphism from G^{op} to H.

Lemma

A right action of G on X is an anti-homomorphism from G to S_X, or equivalently, an action of G^{op} on X.

Thus, we need to fear right actions! They can be treated completetly similarly to left actions. When convenient, we can act on the right, without repercussions or censure!

Henceforth, G acts on X, g, h are elements of G, x, y are element of X.

Definition

The fixed points of g are $\mathtt{Fix}(g) = \{ t \in X | g.t = t \}$.

Definition

The stabilizer of x is the subset $\mathtt{Stab}(x) = \{ u \in G | u.x = x \}$.

Definition

The orbit of x is the subset $\mathtt{Orb}(x) = \{ z \in X | z = u.x \text{ for some } u \in G \}$.

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Example (Trivial action)

If $g.x = x$ always, then

- $\mathtt{Fix}(g) = X$,
- $\mathtt{Stab}(x) = G$,
- $\mathtt{Orb}(x) = \{x\}$.

Example (Linear maps)

For the action $F.\mathtt{v} = F(\mathtt{v})$

- $\mathtt{Fix}(F)$ is the set of eigenvectors of F with eigenvalue 1,
- $\mathtt{Stab}(\mathtt{v})$ is the set of maps that fix the vector $\mathtt{v}$,
- $\mathtt{Orb}(\mathtt{v}) = V$, unless $\mathtt{v} = 0$, in which case it is ???

Example (Groups of symmetries)

$K \subset E^n$ convex, etc.

- $\mathrm{Fix}(F)$ is the set of points in K fixed by F
- $\mathrm{Stab}(\mathrm{p})$ is the set of symmetries of K that fix p,
- $\mathrm{Orb}(\mathrm{p})$ is the set of points that p can get mapped to by a symmetry of K.

For instance, if K is a disc centered at the origin, then

- $\mathrm{Fix}(F)$ is the origin if F is a rotation, the intersection of the line of reflection with K if F is a reflection,
- $\mathrm{Stab}(\mathrm{p})$ is the identity and the reflection through the line through the origin and p, unless p is the origin, in which case it is ???
- $\mathrm{Orb}(\mathrm{p})$ is the circle centered at the origin containing p

Example (Shifting)

- $\mathrm{Fix}(g)$ is the set of all constant sequences, and $\mathrm{Fix}(g^k)$ is the set of all sequences that repeat with period k
- $\mathrm{Stab}((a_j)_{j \in \mathbb{Z}})$ is the identity, unless the sequence repeats with period n, in which case it is $\langle g^n \rangle$
- $\mathrm{Orb}((a_j)_{j \in \mathbb{Z}})$ is the set of all shifts of the sequence!

Example (Cycle diagram)

If $G = \langle \sigma \rangle$ is a cyclic subgroup of S_n, then

- $\mathrm{Fix}(\sigma)$ is the set of all fixed points of the permutation. What is $\mathrm{Fix}(\sigma^k)$?
- $\mathrm{Stab}(i)$ is the set of powers of σ that fix i
- $\mathrm{Orb}(i)$ is the cycle containing i

Example (Conjugation)

Let us consider just the case when G acts on $X = G$ by conjugation, $g.x = gxg^{-1}$.

- $\mathrm{Fix}(g) = \{ x \in X | gxg^{-1} = x \} = \{ g \in G | gx = xg \}$, the set of group elements that commute with g
- $\mathrm{Stab}(x) = \{ g \in G | gxg^{-1} = x \} = \{ g \in G | gx = xg \}$, the set of group elements that commute with x
- $\mathrm{Orb}(x) = \{ gxg^{-1} | g \in G \}$ is the conjugacy class containing x.

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Definition

We say that x, y are G-equivalent if there is a $g \in G$ such that $g.x = y$.

Lemma

G-equivalence is an equivalence relation on X. The equivalence classes are precisely the orbits.

Proof.

Immediate. □

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Lemma

The stabilizer $\mathrm{Stab}(x) = \{ u \in G \mid u.x = x \}$ *is a subgroup of* G.

Proof.

1. $1.x = x$, so $1 \in \mathrm{Stab}(x)$.
2. If $g \in \mathrm{Stab}(x)$, then $g.x = x$, hence $g^{-1}.x = g^{-1}.(g.x) = (g^{-1}g).x = 1.x = x$, so $g^{-1} \in \mathrm{Stab}(x)$.
3. If $g, h \in \mathrm{Stab}(x)$ then $(gh).x = g.(h.x) = g.x = x$, so $gh \in \mathrm{Stab}(x)$.

□

Example

The stabilizer need not be a normal subgroup. For instance, let D_3 act on the vertices of a centered equilateral triangle. Then the stabilizer of 1, the vertex at the x-axis, is the non-normal subgroup generated by the transposition $(2, 3)$.

Theorem

The index of the stabilizer is the size of the orbit, i.e.,

$$[G : \mathrm{Stab}(x)] = |\mathrm{Orb}(x)|.$$

In particular, if G is finite, then $|\mathrm{Orb}(x)| = \frac{|G|}{|\mathrm{Stab}(x)|} < \infty$, even if X is infinite.

Proof

- We will define a bijection F between $\mathrm{Orb}(x)$ and the set of left cosets of $\mathrm{Stab}(x)$.
- Let $y \in \mathrm{Orb}(x)$. Then $g.x = y$ for some $g \in G$.
- Define $F(x) = g\mathrm{Stab}(x)$.
- Is this well defined? If $h.x = y$ as well, then $h^{-1}.y = x$, and $(h^{-1}g.x = x$, so $h^{-1}g \in \mathrm{Stab}(x)$, so $g \in h\mathrm{Stab}(x)$, so $g\mathrm{Stab}(x) = h\mathrm{Stab}(x)$.

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Proof (contd.)

- If $F(y_1) = F(y_2)$ then $g_1.x = y_1$, $g_2.x = y_2$, and $g_1\mathtt{Stab}(x) = g_2\mathtt{Stab}(x)$. Hence $g_1^{-1}g_2 \in \mathtt{Stab}(x)$ so $(g_1^{-1}g_2).x = x$, hence $g_1^{-1}.y_2 = x$, hence $g_1.x = y_2$, so $y_1 = y_2$.
- So F is injective.
- If $g \in G$ and $g\mathtt{Stab}(x)$ is a left coset, put $y = g.x$. Then $G(y) = g\mathtt{Stab}(x)$.
- So F is surjective.
- We have defined a bijection F between $\mathtt{Orb}(x)$ and the set of left cosets of $\mathtt{Stab}(x)$. Since the index $[G : \mathtt{Stab}(x)]$ is the cardinality of the set of left cosets, the theorem is proved.

□

Example

Examples of orbit-stabilizer relation

Theorem (Burnside)

Let the group G be finite, and let G act on the set X. Let r denote the number of orbits. Then

$$r = \frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g) \tag{1}$$

If X is infinite, then r is, as well.
In words: the number of orbits is the average size of the fix-point sets.

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Proof

- We assume that $|X| < \infty$
- Define
$$M = \{ (g,x) | g.x = x \} \subset G \times X$$
- We will count the number of elements in M, in two different ways.
- In "row" g, the number of elements in M are
$$|\{ (g,x) | g.x = x \}| = |\mathrm{Fix}(g)|$$
- In "column" x, the number of elements in M are
$$|\{ (g,x) | g.x = x \}| = |\mathrm{Stab}(x)|$$

Proof (contd.)

- However, by the Orbit-Stabilizer lemma,

$$|\mathrm{Stab}(x)| = \frac{|G|}{|\mathrm{Orb}(x)|}$$

- Since

$$\sum_{g \in G} \text{rowsum } g = \sum_{x \in X} \text{colsum } x$$

we get that

$$\sum_{g \in G} |\mathrm{Fix}(g)| = \sum_{x \in X} \frac{|G|}{|\mathrm{Orb}(x)|}$$

Proof (contd.)

- Let us study the RHS. We pick one element y_i from each of the orbits (thus, a transversal of the G-equivalence relation). Clearly, $|\mathrm{Orb}(x)| = |\mathrm{Orb}(y_i)|$ for all $x \in \mathrm{Orb}(y_i)$, and there are $|\mathrm{Orb}(y_i)|$ such x, so

$$\sum_{x \in X} \frac{|G|}{|\mathrm{Orb}(x)|} = |G| \sum_{x \in X} \frac{1}{|\mathrm{Orb}(x)|} = |G| \sum_{j=i}^{r} \frac{|\mathrm{Orb}(y_j)|}{|\mathrm{Orb}(y_j)|} = |G| r$$

- Putting LHS equal to RHS we get

$$\sum_{g \in G} |\mathrm{Fix}(g)| = |G| r$$

□

Example

Let $\sigma \in S_n$ have c_1 fixed points, c_2 2-cycles et cetera in its disjoint cycle decomposition. Then letting $G = \langle \sigma \rangle$ act on $X = [n]$, the number of orbits is $r = c_1 + c_2 + \dots$. The element $\sigma^k \in G$ fixes $i \in [n]$ iff $\sigma^k(i) = i$; if we take the average as $0 \leq k \leq o(\sigma)$, we should get r.

For instance, if $n = 6$, $\sigma = (1)(2,3)(4,5,6)$ then

k	σ^k	$\mathrm{Fix}(\sigma^k)$
0	$()$	$\{1,2,3,4,5,6\}$
1	$(1)(2,3)(4,5,6)$	$\{1\}$
2	$(1)(2)(3)(4,6,5)$	$\{1,2,3\}$
3	$(1)(2,3)(4)(5)(6)$	$\{1,4,5,6\}$
4	$(1)(2)(3)(4,5,6)$	$\{1,2,3\}$
5	$(1)(2,3)(4,6,5)$	$\{1\}$

The average is $(6+1+3+4+3+1)/6 = 3$, and there are 3 orbits: $\{1\}$, $\{2,3\}$, and $\{4,5,6\}$.

Definition

Let G act on X as before, and let $c : X \to Y$ be a "coloring" of the elements of X; we regard $x \in X$ as having "color" $c(x) \in Y$. Denote the set of coloring by Y^X. Then G acts on Y^X by

$$g.c = c_g \in Y^X$$
$$c_g(x) = c(g.x)$$

Remark

We check that

$$
\begin{aligned}
1.c &= c_1 \\
c_1(x) &= c(1.x) = c(x) \\
1.c &= c \\
(gh).c &= c_{gh} \\
c_{gh}(x) &= c((gh).x) = c(g.(h.x)) \\
g.(h.c) &= g.c_h \\
g.(h.c)(x) &= (g.c_h)(x) = c_h(g.x) = c(h.(g.x)) = ((hg).c)(x)
\end{aligned}
$$

so if we are going to be pedantic, this is a right action. However, as discussed before, that is of no consequence. Just do not be confused!!!!!

Theorem

Let G act on Y^X as before. Suppose that X, Y are finite. Then, with respect to this action, it holds for any $g \in G$ that

$$\mathrm{Fix}(g) = |Y|^n,$$

where n is the number of cycles of the permutation $\phi(g) \in S_x$ induced by the action on G on X.

Proof.

A coloring $c : X \to Y$ is fixed by the induced action of $\phi(g)$ iff it is constant on the cycles of $\phi(g)$. Each cycle can be assigned one out of $|Y|$ colors. □

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Example

Consider the task of coloring the edges of an equilateral triangle; two colorings are equivalent if the colored triangles can be rotated to become equal. Suppose that we can use k colors. Let $g = \langle r \rangle$ be the group of rotations of the triangle. Combining Burnside and the previous thm, we study how the group elements permute the sides of the triangle:

g	$\phi(g)$	$\lvert\mathrm{Fix}(g)\rvert$
1	$(A)(B)(C)$	k^3
r	(A, B, C)	k^1
r^2	(A, C, B)	k^1

Hence, the number of orbits as G acts on the set of inequivalent colorings is

$$\frac{1}{3}(2k^1 + k^3),$$

which, for small values of k is $1, 4, 11, 24, 25$.

Example (Contd.)

If we also consider two colorings equivalent if one can be obtained from the other by flipping the traingle over, then the group is D_3, and the calculation becomes

g	$\phi(g)$	$\lvert\mathrm{Fix}(g)\rvert$
1	$(A)(B)(C)$	k^3
r	(A, B, C)	k^1
r^2	(A, C, B)	k^1
s	$(A, B)(C)$	k^2
sr	$(A, C)(B)$	k^2
sr^2	$(B, C)(A)$	k^2

The number of inequivalent colorings is now

$$\frac{1}{6}(2k^1 + 3k^2 + k^3),$$

which, for small values of k is $1, 4, 10, 20, 35$
It makes sense that with more symmetry, there are fewer non-equivalent colorings.

Definition

Let G act on colorings as before. Define the *cycle index polynomial*

$$Z(G) = \frac{1}{|G|} \sum_{g \in G} Z(g)$$

with $Z(g) = \prod_j a_j^{b_j(g)}$, where the cycle monomial is determined by the cycle type of g, as it acts on X.

Example

The dihedral group D_3 acting on $\{1,2,3\}$ in the natural way has cycle polynomial

$$\frac{1}{6}(a_1^3 + 3a_1a_2 + 2a_3)$$

since there is one permutation with cycle type 1^3, 3 with type $1^1 2^2$, and two of type 3^1.

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Theorem

The number of k-colorings is obtained from the cycle polynomial by substituting $a_i \longrightarrow k$.

Example

Continuing with D_3 acting on $\{1, 2, 3\}$, and colorings thereof: the cycle index is

$$\frac{1}{6}(a_1^3 + 3a_1 a_2 + 2a_3)$$

and the number of k-colorings is

$$\frac{1}{6}(k^3 + 3k^2 + 2k^1)$$

Example

Let D_3 act on $\{1, 2, 3, 4, 5, 6\}$ by acting separately on $\{1, 2, 3\}$ and $\{4, 5, 6\}$, and in addition, any element of D_3 which is orientation-reversing interchanges $\{1, 2, 3\}$ with the corresponding element in $\{4, 5, 6\}$. Then the action induces the permutations

g	$\hat{g}$	$Z(\hat{g})$
$()$	$()$	a_1^6
(123)	$(123)(456)$	a_3^2
(132)	$(132)(465)$	a_3^2
(12)	$(15)(24)$	$a_1^2 a_2^2$
(13)	$(16)(34)$	$a_1^2 a_2^2$
(23)	$(26)(35)$	$a_1^2 a_2^2$

with cycle polynomial $\frac{1}{6}(a_1^6 + 3a_1^2 a_2^2 + 2a_3^2)$ and number of non-equivalent k-colorings $\frac{1}{6}(k^6 + 3k^4 + 2k^2)$. For $k = 2$ we get 20 2-colorings.

Polya theory

There are various further developments and refinements of this idea, using so-called weighted cycle index. We give one example: if we want to count the number of 2-colorings that use precisely r of the first color, we take the cycle index and substitute $a_j \longrightarrow (1 + t^j)$ and expand the result; the coefficient of t^r is the answer.

Example

Consider again D_3 acting on $\{1,2,3,4,5,6\}$ by acting separately on $\{1,2,3\}$ and $\{4,5,6\}$, and in addition, any element of D_3 which is orientation-reversing interchanges $\{1,2,3\}$ with the corresponding element in $\{4,5,6\}$. As we have seen, the cycle polynomial is

$$\frac{1}{6}(a_1^6 + 3a_1^2a_2^2 + 2a_3^2).$$

Substituting $a_j \mapsto (1+t^j)$ we get

$$\frac{1}{6}(t+1)^6 + \frac{1}{2}\left(t^2+1\right)^2(t+1)^2 + \frac{1}{3}\left(t^3+1\right)^2 = t^6 + 2\,t^5 + 4\,t^4 + 6\,t^3 + 4\,t^2 + 2\,t + 1$$

so there is 2 ways of coloring the vertices red or blue, with 2 red and 4 blue vertices.

Example

Let G be the rigid symmetry group of the cube, and let it act on vertices. The cycle index is then

$$\frac{1}{24}(a_1^8 + 9a_2^4 + 6a_4^2 + 8a_1^2a_3^2)$$

which after substituting $a_j \longrightarrow (1+t)$, and expanding, becomes

$$1 + t + 3t^2 + 7t^4 + 3t^5 + 3t^6 + t^7 + t^8.$$

Thus, there are 3 in-equivalent ways of coloring 2 corners red and 6 blue. Substituting $a_j \longrightarrow k$ we get

$$\frac{1}{24}(k^8 + 17k^4 + 6k^2)$$

which gives the number of ways of coloring the corners with k colors.