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April 29, 2021

1 Notation and Definitions

- **R** is the set of all real numbers.
- **Q** is the set of all rational numbers.
- **C** is the set of all complex numbers.
- **Z** is the set of all integers.
- $\mathbf{N} = \{0, 1, 2, \ldots\}$ is the set of all natural numbers (including 0).

For $z = x + iy \in \mathbf{C}$, $x, y \in \mathbf{R}$,

$$\operatorname{Re} z = x, \quad \operatorname{Im} z = y, \quad |z| = \sqrt{x^2 + y^2}.$$

1.1 Continuity and Differentiability

- One-sided limits:
$$u(x^\pm) = \lim_{x \to x^\pm} u(x).$$
- One-sided derivatives:
$$D^\pm u(x) = \lim_{h \to 0^\pm} \frac{u(x+h) - u(x)}{h}.$$
- $C(I)$: The set of all continuous functions on a set I.
- $C^m(I)$: The set of all continuously differentiable (up to order m) functions on a set I.

A function $u\colon I \to \mathbf{C}$ on an interval I is called piecewise continuous if...

- I is finite and there are a finite number of points such that u is continuous everywhere on I except for at these points. Moreover, if $c \in I$ is a point where u is discontinuous, the limits
$$\lim_{I \ni x \to c^-} u(x) \quad \text{and} \quad \lim_{I \ni x \to c^+} u(x)$$
exist (only $u(c^-)$ or $u(c^+)$ if points on the boundary of I).
- I is infinite and there a finite number of exception points (as in the finite case) in each finite sub-interval of I.

1.2 Function Spaces

A normed linear space is a linear space V endowed with a norm $\|\cdot\|\colon V \to [0,\infty[$ such that

$$\text{(i) } \|u\| \geq 0 \qquad \text{(ii) } \|\alpha u\| = |\alpha|\|u\|,\ \alpha \in \mathbf{C} \qquad \text{(iii) } \|u+v\| \leq \|u\| + \|v\|.$$

An inner product $\langle \cdot,\ \cdot \rangle : V \times V \to \mathbf{C}$ on a vector space V satisfies

$$\text{(i) } \langle u,\ v\rangle = \overline{\langle v,\ u\rangle} \qquad \text{(ii) } \langle u+v,\ w\rangle = \langle u,\ w\rangle + \langle v,\ w\rangle \qquad \text{(iii) } \langle \alpha u,\ v\rangle = \alpha \langle u,\ v\rangle$$

$$\text{(iv) } \langle u,\ u\rangle \geq 0 \qquad \text{(v) } \langle u,\ u\rangle = 0 \quad \Leftrightarrow \quad u = 0.$$

In an inner product space, we use $\|u\| = \sqrt{\langle u,\ u\rangle}$ as the norm.

Sequence Spaces

The sequence spaces l^p, $1 \leq p \leq \infty$, consists of sequences $(x_0, x_1, x_2, x_3, \ldots)$, $x_i \in \mathbf{C}$, such that the norm

$$\|x\|_{l^p} = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

or

$$\|x\|_{l^\infty} = \sup_{k\geq 0} |x_k| < \infty, \quad p = \infty.$$

Sometimes we write $l^p(\mathbf{N})$. The spaces $l^p(\mathbf{Z})$ are defined analogously. Only l^2 is an inner product space with

$$\langle x,\ y\rangle = \sum_{k=0}^{\infty} x_k \overline{y_k}, \quad x, y \in l^2.$$

Lebesgue Spaces (integrable functions)

The space $L^1(a,b)$ of absolutely integrable functions $u\colon]a,b[\to \mathbf{C}$ with norm

$$\|f\|_{L^1(a,b)} = \int_a^b |f(x)|\, dx < \infty.$$

The space $L^2(a,b)$ consists of all "square integrable" functions with the norm

$$\|f\|_{L^2(a,b)} = \left(\int_a^b |f(x)|^2\, dx\right)^{1/2} < \infty,$$

which is an inner product space with

$$\langle f,\ g\rangle = \int_a^b f(x)\overline{g(x)}\, dx.$$

The space $L^\infty(a,b)$ of bounded functions with norm

$$\|f\|_{L^\infty(a,b)} = \sup_{a\leq x\leq b} |f(x)| < \infty.$$

Note that $a = -\infty$ and/or $b = \infty$ is allowed (so we might write $L^p(\mathbf{R})$). Sometimes we write $\|f\|_p$ instead of $\|f\|_{L^p(a,b)}$.

Spaces of Piecewise Functions

- $E[a, b]$ (or E): The linear space of all piecewise continuous functions on an interval $[a, b]$.
- $E'[a, b]$ (or E'): The linear space of those $u \in E[a, b]$ such that $D^- u(x)$ exists for $a < x \le b$ and that $D^+ u(x)$ exists for $a \le x < b$.
- $G(\mathbf{R})$ (or G): The linear space of all piecewise continuous functions on $\mathbf{R}$ that are absolutely integrable on $\mathbf{R}$.

1.3 Special Functions

- **Heaviside function**:

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \ge 0. \end{cases}$$

- **Signum function**:

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Discrete Functions

- **Discrete Heaviside function**:

$$H[k] = \begin{cases} 0, & k < 0, \\ 1, & k \ge 0. \end{cases}$$

- **Discrete impulse function**:

$$\delta[k] = \begin{cases} 0, & k \ne 0, \\ 1, & k = 0. \end{cases}$$

- **Binomial coefficient functions**:

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{(n-k)!\,k!}, & k = 0, 1, 2, \dots, \\ 0, & k > n. \end{cases}$$

Convolutions (on R)

The convolution $u * v \colon \mathbf{R} \to \mathbf{C}$ of two functions $u \colon \mathbf{R} \to \mathbf{C}$ and $v \colon \mathbf{R} \to \mathbf{C}$ is defined by

$$(u * v)(x) = \int_{-\infty}^{\infty} u(t) v(x - t)\, dt, \quad x \in \mathbf{R},$$

whenever this integral exists. If $u, v \in L^1(\mathbf{R})$, then $u * v \in L^1(\mathbf{R})$. If one functions is also bounded, then $u * v$ is continuous and bounded.
Suppose that $u, v, w \in G(\mathbf{R})$ and one function in each convolution is bounded. Then the convolution has the following properties.

- Associative: $((u * v) * w)(x) = (u * (v * w))(x)$.
- Distributive: $((u + v) * w)(x) = (u * w)(x) + (v * w)(x)$.
- Commutative: $(u * v)(x) = (v * u)(x)$.

Convolutions (on Z)

- For $u, v\colon \mathbf{Z} \to \mathbf{C}$, the **discrete convolution** $u * v$ is

$$(u * v)[n] = \sum_{k=-\infty}^{\infty} u[k]v[n-k], \quad n \in \mathbf{Z},$$

  whenever this series exists.

- For $u, v\colon \mathbf{N} \to \mathbf{C}$, the **unilateral (or one-sided) discrete convolution** $u * v$ is

$$(u * v)[n] = \sum_{k=0}^{n} u[k]v[n-k], \quad n = 0, 1, 2, \ldots$$

1.4 Inequalities

- **The Cauchy-Schwarz inequality**: If $u, v \in V$ and V is an inner product space, then

$$|\langle u,\ v\rangle| \leq \|u\|\|v\|.$$

- **Bessel's inequality**: Let V be an inner product space, let $v \in V$ and let $\{e_1, e_2, \ldots\}$ be an ON system in V. Then

$$\sum_{k=1}^{\infty} |\langle v,\ e_k\rangle|^2 \leq \|v\|^2.$$

  This implies the **Riemann-Lebesgue lemma** for inner product spaces:

$$\lim_{n\to\infty} \langle v,\ e_n\rangle = 0.$$

- **The triangle inequality**: In a normed space V,

$$|\|u\| - \|v\|| \leq \|u + v\| \leq \|u\| + \|v\|.$$

- **Young's inequality** $(r = p = q = 1)$:

$$\|u * v\|_{L^1(\mathbf{R})} \leq \|u\|_{L^1(\mathbf{R})}\|v\|_{L^1(\mathbf{R})}.$$

  and

$$\|u * v\|_{l^1(\mathbf{Z})} \leq \|u\|_{l^1(\mathbf{Z})}\|v\|_{l^1(\mathbf{Z})}.$$

1.5 Convergence of Sequences

Let $u_1, u_2, \ldots$ be a sequence in a normed space V. We say that $u_n \to u$ for some $u \in V$ if $\|u_n - u\| \to 0$ as $n \to \infty$. This is called **strong convergence** or **convergence in norm**.

Convergence of Functions

- **Pointwise convergence**: We say that $u_k \to u$ pointwise on I as $k \to \infty$ if
$$\lim_{k\to\infty} u_k(x) = u(x)$$
for every $x \in I$. We often refer to u as the *limiting function*.
- **Uniform convergence**: We say that $u_k \to u$ uniformly on $[a, b]$ as $k \to \infty$ if
$$\lim_{k\to\infty} \|u_k - u\|_{L^\infty(a,b)} = 0.$$

Weierstrass' M-test: If $I \subset \mathbf{R}$ and M_k, $k = 1, 2, \ldots$, are constants such that $|u_k(x)| \le M_k$ for $x \in I$, then
$$\sum_{k=1}^{\infty} M_k < \infty \quad \Rightarrow \quad \sum_{k=1}^{\infty} u_k(x) \text{ converges uniformly on } I.$$

If:

- $u_0, u_1, u_2, \ldots$ are continuous functions on $[a, b]$
- and $u(x) = \sum_{k=0}^{\infty} u_k(x)$ is uniformly convergent for $x \in [a, b]$,

then

- the series u is a continuous function on $[a, b]$,
- we can exchange the order of summation and integration:
$$\int_c^d u(x)\,dx = \int_c^d \left(\sum_{k=0}^{\infty} u_k(x)\right)\,dx = \sum_{k=0}^{\infty} \int_c^d u_k(x)\,dx, \quad \text{for } a \le c < d \le b,$$
- and if in addition $\sum_{k=0}^{\infty} u_k'(x)$ converges uniformly on $[a, b]$, then
$$u'(x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} u_k(x)\,dx\right) = \sum_{k=0}^{\infty} \frac{d}{dx} u_k(x) = \sum_{k=0}^{\infty} u_k'(x), \quad x \in [a, b].$$

1.6 Integration Theory

The **principal value** integral is defined by
$$\text{P. V.} \int_{-\infty}^{\infty} u(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} u(x)\,dx.$$

- If $F(x) = \int_{-\infty}^{\infty} f(x, y)\,dy$ exists for every $x \in I$ and
$$\sup_{x\in I} \left| \int_{-R}^{R} f(x, y)\,dy - F(x) \right| \to 0, \text{ as } R \to \infty,$$
then we call the integral defining $F(x)$ uniformly convergent on I.

- **Dominated convergence**:

  If:

  - $f\colon \mathbf{R}^2 \to \mathbf{C}$,
  - $F(x) = \displaystyle\int_{-\infty}^{\infty} f(x,y)\,dy$ exists for all x,
  - there is a $g \in L^1(\mathbf{R})$ such that $|f(x,y)| \le g(y)$ for all $x, y \in \mathbf{R}$,

  then $\displaystyle\int_{-\infty}^{\infty} f(x,y)\,dy$ converges uniformly on $\mathbf{R}$.

- **Continuity**: If $f\colon \mathbf{R}^2 \to \mathbf{C}$ is continuous on $[c,d] \times [a,R]$. Then

  - $F_R(x) = \displaystyle\int_a^R f(x,y)\,dy$ is continuous on $[c,d]$
  - and if in addition f is continuous on $[c,d] \times [a,\infty[$ and $F(x) = \displaystyle\int_a^{\infty} f(x,y)\,dy$ converges uniformly (on $[c,d]$), then F is continuous.

- **Order of integration**: If $f\colon \mathbf{R}^2 \to \mathbf{C}$ is continuous on $[c,d] \times [a,\infty[$ and $F(x)$ converges uniformly (on $[c,d]$), then

$$\int_c^d \left(\int_a^{\infty} f(x,y)\,dy \right) dx = \int_a^{\infty} \left(\int_c^d f(x,y)\,dx \right) dy.$$

- Note that we can let $a = -\infty$ in the previous theorems by exchanging $[a,R]$ by $[-R,R]$ and consider the principal values.

- **Leibniz's rule**: If

  - $f\colon \mathbf{R}^2 \to \mathbf{C}$ and $f'_x(x,y)$ exist and are continuous,
  - $F(x) = \displaystyle\int_{-\infty}^{\infty} f(x,y)\,dy$ is convergent for every x,
  - and $\displaystyle\int_{-\infty}^{\infty} f'_x(x,y)\,dy$ is uniformly convergent,

  then

$$F'(x) = \frac{d}{dx} \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\infty}^{\infty} f'_x(x,y)\,dy.$$

2 Fourier Series

For $u \in L^1(-\pi, \pi)$:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\cos kx\, dx \quad \text{and} \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\sin kx\, dx \quad \text{or} \quad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\, dx$$

are the Fourier coefficients (real or complex) for u. The series

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\big(a_k \cos kx + b_k \sin kx\big) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

is called the **Fourier series** of the function u (real or complex). We write $u(x) \sim S(x)$.
Note that:

- if u is even, then $b_k = 0$ for $k = 1, 2, 3, \dots$;
- if u is odd, then $a_k = 0$ for $k = 1, 2, 3, \dots$.

If u is a T-periodic function, we define $\Omega = \frac{2\pi}{T}$. The real Fourier series of u is then given by

$$u(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}\big(a_k \cos k\Omega x + b_k \sin k\Omega x\big),$$

where

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} u(x)\cos k\Omega x\, dx \quad \text{and} \quad b_k = \frac{2}{T}\int_{-T/2}^{T/2} u(x)\sin k\Omega x\, dx.$$

The complex Fourier series is given by

$$u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ik\Omega x}, \quad \text{where } c_k = \frac{1}{T}\int_{-T/2}^{T/2} u(x)e^{-ik\Omega x}\, dx.$$

Sometimes we denote $c_k = \widehat{u}[k]$.

2.1 Parseval's identity

- **Parsevals's identity**:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|^2\, dx = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad \text{or} \quad \frac{1}{\pi}\int_{-\pi}^{\pi} |u(x)|^2\, dx = \frac{|a_0|^2}{2} + \sum_{k=1}^{\infty}\big(|a_k|^2 + |b_k|^2\big),$$

  where $u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ (or the real counterpart).

- **Parseval's generalized identity**:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\overline{v(x)}\, dx = \sum_{k=-\infty}^{\infty} c_k \overline{d_k},$$

  where $u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ and $v(x) \sim \sum_{k=-\infty}^{\infty} d_k e^{ikx}$.

2.2 Convergence

Kernels

- The **Dirichlet kernel**: $D_n(x) = \sum_{k=-n}^{n} e^{ikx},\ x \in \mathbf{R},\ n = 1, 2, 3, \ldots$
- The **Fejér kernel**: $F_n(x) = \frac{1}{n+1} \sum_{l=0}^{n} \sum_{k=-l}^{l} e^{ikx} = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikx}, \quad n = 0, 1, 2, \ldots$

2.3 Convergence Results

- If $u \in L^1(-\pi, \pi)$, then u has a Fourier series.
- Let $u \in E'$. Then
$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} \to \frac{u(x^+) + u(x^-)}{2}, \quad x \in [-\pi, \pi].$$
- If $u \in E$ and $D^{\pm}u(x)$ exists, then
$$\lim_{n\to\infty} S_n(x) = \frac{u(x^+) + u(x^-)}{2}.$$
- If $\sum_{k=-\infty}^{\infty} |c_k| < \infty$, then $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ converges uniformly.
- If $u \in E$, then the Fejér means $\overline{S_n}(x) \to \frac{u(x^+) + u(x^-)}{2}$.
- If $u, v \in E$ and $\widehat{u}[k] = \widehat{v}[k]$, $k \in \mathbf{Z}$, then $u(x) = v(x)$ whenever u and v are continuous at x.
- If $u' \in E$, u is continuous and $u(-\pi) = u(\pi)$, then $S_n(x)$ converges uniformly to $u(x)$.
- If $u' \in E$ and u is continuous on $[a, b] \subset]-\pi, \pi[$, then $S_n(x)$ converges uniformly on $[a, b]$.
- If $u \in E$ is continuous and $u(-\pi) = u(\pi)$, then $\overline{S_n}(x)$ converges uniformly to $u(x)$.

The statement $u' \in E$ does not mean that $u'(x)$ exists everywhere, but that there exists a $v \in E$ such that $v(x) = u'(x)$ when $u'(x)$ exists and that u' exists everywhere except at a finite number of points in $[a, b]$.

2.4 General Fourier Series

- For a given ON system, the complex numbers $\langle v,\ e_k \rangle$, $k = 1, 2, \ldots$, are called the **generalized Fourier coefficients** of v.
- If $W = \{e_1, e_2, \ldots\}$ is an ON system in V, then W is closed if and only if **Parseval's identity** holds:
$$\sum_{k=1}^{\infty} |\ \langle v,\ e_k \rangle\ |^2 = \|v\|^2, \quad v \in V,$$
or if $a_k = \langle u,\ e_k \rangle$ and $b_k = \langle v,\ e_k \rangle$, then
$$\langle u,\ v \rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}.$$

2.5 Rules for Fourier Coefficients

Let $u, v \in E$ be periodic with period $T > 0$ and define $\Omega = 2\pi/T$.

Table 1: Rules for Fourier Coefficients

Function	Fourier coefficient	Notes
$c_1 u(x) + c_2 v(x)$	$c_1 \widehat{u}[n] + c_2 \widehat{v}[n]$	
$(u * v)_T(x)$	$\widehat{u}[n]\, \widehat{v}[n]$	periodic convolution†
$u(x)v(x)$	$(\widehat{u} * \widehat{v})[n]$	
$e^{im\Omega x} u(x)$	$\widehat{u}[n-m]$	$m \in \mathbf{Z}$
$u(x-a)$	$e^{-in\Omega a}\, \widehat{u}[n]$	$a \in \mathbf{R}$
$u(ax)$	$\widehat{u}[n]$	period T/a, $a > 0$
$u(-x)$	$\widehat{u}[-n]$	
$\overline{u(x)}$	$\overline{\widehat{u}[-n]}$	
$u'(x)$	$in\Omega \widehat{u}[n]$	
$u^{(k)}(x)$	$(i\Omega n)^k \widehat{u}[n]$	$k = 1, 2, \ldots$

† $(u * v)_T(x) = \dfrac{1}{T} \displaystyle\int_0^T u(x-t)v(t)\, dt.$

3 The Fourier Transform

The **Fourier transform** of a function $u\colon \mathbf{R} \to \mathbf{C}$ given by

$$U(\omega) = \widehat{u}(\omega) = \mathcal{F}\,u(\omega) = \int_{-\infty}^{\infty} u(x)e^{-i\omega x}\,dx, \quad \omega \in \mathbf{R},$$

when this integral exists.

- If $u \in L^1(\mathbf{R})$ then $\mathcal{F}\,u(\omega)$ exists for all $\omega \in \mathbf{R}$ and

$$\|\mathcal{F}\,u\|_\infty \leq \|u\|_{L^1(\mathbf{R})}.$$

- For $u \in G$, the Fourier transform $\mathcal{F}\,u$ is uniformly continuous on $\mathbf{R}$.
- **The Riemann-Lebesgue lemma**: For $u \in G$ we have $\mathcal{F}\,u(\omega) \to 0$ as $|w| \to \infty$.

3.1 Convergence

Kernels

- The **Dirichlet kernel** (on $\mathbf{R}$):

$$D_R(x) = \frac{\sin(Rx)}{\pi x}, \quad x \neq 0,$$

  and $D_R(0) = R/\pi$.
- The **Fejér kernel** (on $\mathbf{R}$):

$$F_M(t) = \frac{1}{2\pi}\int_{-M}^{M}\left(1 - \frac{|\omega|}{M}\right)e^{i\omega t}\,d\omega = \frac{1-\cos Mx}{\pi M x^2} = \frac{M}{2\pi}\left(\frac{\sin(Mx/2)}{Mx/2}\right)^2,$$

  where the last two equalities assumes that $x \neq 0$.

Inversion

- If $u \in G(\mathbf{R})$ and $D^{\pm}u(x)$ exists, then

$$\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R}\mathcal{F}\,u(\omega)e^{i\omega x}\,d\omega = \frac{u(x^+)+u(x^-)}{2}.$$

- If $u \in G(\mathbf{R})$, then

$$\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R}\mathcal{F}\,u(\omega)\left(1-\frac{|\omega|}{R}\right)e^{i\omega x}\,d\omega = \frac{u(x^+)+u(x^-)}{2}.$$

- If $u \in G(\mathbf{R})$, then

$$\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R}\mathcal{F}\,u(\omega)e^{i\omega x}\,d\omega = \frac{u(x^+)+u(x^-)}{2},$$

  whenever the limit exists.
- **Uniqueness**: If $u, v \in G(\mathbf{R})$ and $\mathcal{F}\,u(\omega) = \mathcal{F}\,v(\omega)$ for every $\omega \in \mathbf{R}$, then $u(x) = v(x)$ for every $x \in \mathbf{R}$ where both u and v are continuous.

3.2 Special Rules

- If $u, U \in G(\mathbf{R})$ and $U(\omega) = \mathcal{F}(u)(\omega)$, then

$$\mathcal{F}^{-1}(U)(x) = \frac{1}{2\pi}\,\mathcal{F}\big((\mathcal{F}\,u)(-\omega)\big)(x) \quad \text{and} \quad \mathcal{F}(\mathcal{F}\,u(\omega))(x) = 2\pi u(-x),$$

  for every x where u is continuous and $D^{\pm}u(x)$ exist.

- If $u, v \in G(\mathbf{R})$ such that $uv, \mathcal{F}\,u, \mathcal{F}\,v \in G(\mathbf{R})$, then

$$\mathcal{F}(uv)(\omega) = \frac{1}{2\pi}(\mathcal{F}(u) * \mathcal{F}(v))(\omega).$$

3.3 Plancherel's Theorem

- If $u \in G(\mathbf{R}) \cap L^2(\mathbf{R})$, then $\mathcal{F}\,u \in L^2(\mathbf{R})$.
- **Parseval's formula**: If $u, v \in G(\mathbf{R}) \cap L^2(\mathbf{R})$, then

$$\int_{-\infty}^{\infty} u(x)\,\overline{v(x)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}\,u(\omega)\,\overline{\mathcal{F}\,v(\omega)}\,d\omega.$$

3.4 Rules for the Fourier Transform

Let $u, v \in G(\mathbf{R})$ with $U(\omega) = \mathcal{F}\,u(\omega)$ and $V(\omega) = \mathcal{F}\,v(\omega)$.

Table 2: Rules for the Fourier transform

Function	Fourier transform	Notes
$c_1u(x) + c_2v(x)$	$c_1U(\omega) + c_2V(\omega)$	
$(u * v)(x)$	$U(\omega)V(\omega)$	
$e^{iax}u(x)$	$U(\omega - a)$	$a \in \mathbf{R}$
$u(x)\cos ax$	$\dfrac{U(\omega - a) + U(\omega + a)}{2}$	$a \in \mathbf{R}$
$u(x)\sin ax$	$\dfrac{U(\omega - a) - U(\omega + a)}{2i}$	$a \in \mathbf{R}$
$u(x - x_0)$	$e^{-ix_0\omega}U(\omega)$	$x_0 \in \mathbf{R}$
$u(ax)$	$\dfrac{1}{\lvert a\rvert}\,U\left(\dfrac{\omega}{a}\right)$	$a \in \mathbf{R}$, $a \neq 0$
$\overline{u(x)}$	$\overline{U(-\omega)}$	
$u'(x)$	$i\omega U(\omega)$	$u \in C(\mathbf{R})$, $u' \in G$
$u^{(k)}(x)$	$(i\omega)^kU(\omega)$	$u^{(k)} \in G(\mathbf{R})$
$x^mu(x)$	$i^mU^{(m)}(\omega)$	$x^mu(x) \in G$, $m = 1, 2, 3, \ldots$

3.5 Fourier Transforms

Table 3: Fourier transforms

Function	Fourier transform	Notes
e^{-ax^2}	$\sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}$	$a > 0$
$e^{-a\|x\|}$	$\frac{2a}{a^2+\omega^2}$	$a > 0$
$\operatorname{sgn}(x)e^{-a\|x\|}$	$\frac{-2i\omega}{a^2+\omega^2}$	$a > 0$
$H(x)e^{-ax}$	$\frac{1}{a+i\omega}$	$\operatorname{Re} a > 0$
$H(-x)e^{ax}$	$\frac{1}{a-i\omega}$	$\operatorname{Re} a > 0$
$\frac{1}{a^2+x^2}$	$\frac{\pi}{a}e^{-a\|\omega\|}$	$a > 0$
$H(x+a) - H(x-a)$	$\frac{2\sin a\omega}{\omega}$	$a > 0$
$\operatorname{sgn}(x)\big(H(x+a) - H(x-a)\big)$	$\frac{2(1-\cos a\omega)}{i\omega}$	$a > 0$
$(a - \|x\|)\big(H(x+a) - H(x-a)\big)$	$\frac{2(1-\cos a\omega)}{\omega^2}$	$a > 0$
$\frac{1-\cos at}{t^2}$	$\pi(a - \|\omega\|)\big(H(\omega+a) - H(\omega-a)\big)$	$a > 0$

4 The (unilateral) Laplace Transform

The Laplace transform of $u\colon [0,\infty[\to \mathbf{C}$ is given by

$$\mathcal{L}\,u(s) = \int_0^\infty u(t)e^{-st}\,dt,$$

for those $s = \sigma + i\omega \in \mathbf{C}$, $\sigma, \omega \in \mathbf{R}$, where this integral is convergent.

- **Exponential growth**: A piecewise continuous $u\colon [0,\infty[$ is of exponential growth (of order a) if there exists constants $a > 0$ and $K > 0$ such that $|u(t)| \le Ke^{at}$ for $t \ge 0$. The set of all such functions will be denoted by X_a.
- **Existence of $\mathcal{L}\,u(s)$**: If $u \in X_a$ for some $a > 0$, then the Laplace transform $\mathcal{L}\,u(s)$ exists (at least) for $\operatorname{Re} s > a$.
- $\mathcal{L}\,u(s) \to 0$ as $\mathbf{R} \ni s \to \infty$.
- $\mathcal{L}\,u(s)$ converges uniformly for $\operatorname{Re} s > a$.
- $\mathcal{L}\,u(s)$ is analytic for $\operatorname{Re} s > a$.
- **Periodicity**: If there exists $T > 0$ such that $u(t+T) = u(t)$ for every $t \ge 0$, then

$$\mathcal{L}\,u(s) = \frac{1}{1-e^{-sT}} \int_0^T u(\tau)e^{-s\tau}\,d\tau.$$

4.1 Inversion

- If $u \in X_a$ has right- and lefthand limits at a point $t > 0$, then

$$\lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \mathcal{L}\,u(\sigma + i\omega)e^{\sigma t}e^{i\omega t}\,d\omega = \frac{u(t^+) + u(t^-)}{2},$$

  where the vertical line $\operatorname{Re} z = \sigma$ is contained in the region of convergence of $\mathcal{L}\,u(s)$
- If $u, v \in X_a$ and $\mathcal{L}\,u(s) = \mathcal{L}\,v(s)$ on some vertical line $\operatorname{Re} s = \sigma$, then $u(t) = v(t)$ for all t where u and v are continuous.

4.2 Limit Theorems

- **Final value theorem**:
  If $u\colon [0,\infty[\to \mathbf{C}$ is bounded and $\lim_{t\to\infty} u(t) = A$, then $A = \lim_{\mathbf{R}\ni s\to 0^+} s\,\mathcal{L}\,u(s)$.
- **Initial value theorem**:
  If $u\colon [0,\infty[\to \mathbf{C}$ belongs to X_b and $\lim_{t\to 0^+} u(t) = a$, then $a = \lim_{\mathbf{R}\ni s\to\infty} s\,\mathcal{L}\,u(s)$.

4.3 Rules for the Laplace Transform

Let $U(s) = \mathcal{L}\, u(t)$, $\sigma > \sigma_u$ and $V(s) = \mathcal{L}\, v(t)$, $\sigma > \sigma_v$.

Table 4: Rules for Laplace transforms

Function	Unilateral Laplace transform	Region of convergence
$c_1 u(t) + c_2 v(t)$	$c_1 U(s) + c_2 V(s)$	$\sigma > \max\{\sigma_u, \sigma_v\}$
$(u * v)(t)$	$U(s)V(s)$	unilateral conv.†; $\sigma > \max\{\sigma_u, \sigma_v\}$
$e^{at}u(t)$	$U(s-a)$	$\sigma > \sigma_u + \operatorname{Re} a$
$u(t-t_0)H(t-t_0)$	$e^{-t_0 s}U(s)$	$\sigma > \sigma_u$, $t_0 > 0$
$u(at)$	$\frac{1}{a} U\left(\frac{s}{a}\right)$	$\sigma > a\sigma_u$, $a > 0$
$\overline{u(t)}$	$\overline{U(\overline{s})}$	$\sigma > \sigma_u$
$u'(t)$	$sU(s) - u(0)$	$\sigma > \sigma_u$
$u^{(n)}(t)$	$s^n U(s) - s^{n-1}u(0) - \cdots$ $\cdots - s u^{(n-2)}(0) - u^{(n-1)}(0)$	$\sigma > \max\{\sigma_u, \sigma_{u'}, \ldots, \sigma_{u^{(n-1)}}\}$
$\int_0^t u(\tau)\, d\tau$	$\frac{U(s)}{s}$	$\sigma > \max\{\sigma_u, 0\}$
$t^m u(t)$	$(-1)^m U^{(m)}(s)$	$\sigma > \sigma_u$

† $(u * v)(t) = \int_0^t u(\tau)v(t-\tau)\, d\tau$.

4.4 Laplace Transforms

Table 5: Laplace transforms

Function	Unilateral Laplace transform	Region of convergence
$H(t) = 1$	$\frac{1}{s}$	$\sigma > 0$
t	$\frac{1}{s^2}$	$\sigma > 0$
t^m	$\frac{m!}{s^{m+1}}$	$\sigma > 0$ $m = 1, 2, 3, \ldots$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$	$\sigma > 0$ $a > 0$
e^{at}	$\frac{1}{s-a}$	$\sigma > \operatorname{Re} a$
$t^m e^{at}$	$\frac{m!}{(s-a)^{m+1}}$	$\sigma > \operatorname{Re} a$ $m = 1, 2, 3, \ldots$
$\cos at$	$\frac{s}{s^2+a^2}$	$\sigma > \lvert \operatorname{Im} a \rvert$
$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$\sigma > \lvert \operatorname{Im} a \rvert$
$\sin at$	$\frac{a}{s^2+a^2}$	$\sigma > \lvert \operatorname{Im} a \rvert$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2}$	$\sigma > \lvert \operatorname{Im} a \rvert$
$\frac{\sin at}{t}$	$\arctan\left(\frac{a}{s}\right)$	$\sigma > \lvert \operatorname{Im} a \rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$\sigma > \lvert \operatorname{Re} a \rvert$
$\sinh at$	$\frac{a}{s^2-a^2}$	$\sigma > \lvert \operatorname{Re} a \rvert$
$J_0(at)$	$\frac{1}{\sqrt{a^2+s^2}}$	$\sigma > \lvert \operatorname{Im} a \rvert$

5 The (unilateral) Z Transform

The **Z transform** of a sequence $u[k]$, $k = 0, 1, 2, \ldots$, is defined by

$$\mathcal{Z}(u)(z) = \sum_{k=0}^{\infty} u[k]\, z^{-k},$$

for those $z = x + iy \in \mathbf{C}$, $x, y \in \mathbf{R}$, where this series is absolutely convergent.

- **Existence of $\mathcal{Z}\, u(z)$**: For a sequence $u[k]$, $k = 0, 1, 2, \ldots$, the Z transform $\mathcal{Z}\, u(z)$ has a region of convergence R such that $\mathcal{Z}\, u(z)$ is absolutely (uniformly) convergent for $|z| > R$ and divergent for $|z| < R$. It is possible that $R = 0$ or $R = \infty$.
- **Inversion**: If $U(z) = \mathcal{Z}\, u(z)$, then

  $$u[k] = \frac{1}{2\pi i} \oint_{\gamma} z^{k-1} U(z)\, dz, \quad k = 0, 1, 2, \ldots,$$

  where γ is a closed curve (inside $|z| > R_u$) counterclockwise around the origin.
- **Uniqueness**: If $\mathcal{Z}\, u(z) = \mathcal{Z}\, v(z)$ for all $|z| > R$ for some $R > 0$, then $u[k] = v[k]$ for $k = 0, 1, 2, \ldots$.
- **Initial value theorem**: If there's an $R > 0$ such that $\mathcal{Z}\, u(z)$ exists for $|z| > R$, then

  $$\lim_{|z| \to \infty} \mathcal{Z}\, u(z) = u[0].$$

5.1 Rules for the Z Transform

Let $U(z) = \mathcal{Z}(u[k])(z)$, $|z| > R_u$ and $V(z) = \mathcal{Z}(v[k])(z)$, $|z| > R_v$.

Table 6: Rules for Z transforms

Function	Unilateral Z transform transform	Region of convergence
$c_1 u[k] + c_2 v[k]$	$c_1 U(z) + c_2 V(z)$	$\|z\| > \max\{R_u, R_v\}$
$(u * v)[k]$	$U(z)V(z)$	unilateral conv.†; $\|z\| > \max\{R_u, R_v\}$
$a^k u[k]$	$U\left(\frac{z}{a}\right)$	$\|z\| > \|a\| R_u$, $a \in \mathbf{C} \setminus \{0\}$
$u[k-m]H[k-m]$	$z^{-m}U(z)$	$\|z\| > R_u$, $m = 1, 2, 3, \ldots$
$u[k-m]$	$z^{-m}U(z) + z^{-m+1}u[-1] + \cdots$ $\cdots + z^{-1}u[-m+1] + u[-m]$	$\|z\| > R_u$, $m = 1, 2, 3, \ldots$
$u[k+m]$	$z^m U(z) - z^m u[0] + \cdots$ $\cdots - z^2 u[m-2] - z u[m-1]$	$\|z\| > R_u$, $m = 1, 2, 3, \ldots$
$\overline{u[k]}$	$\overline{U(\overline{z})}$	$\|z\| > R_u$
$\sum_{l=0}^{k} u[l]$	$\frac{z}{z-1} U(z)$	$\|z\| > \max\{R_u, 1\}$
$k^m u[k]$	$\left(-z\frac{d}{dz}\right)^m U(z)$	$\|z\| > R_u$

† $(u * v)[k] = \sum_{l=0}^{k} u[l]v[k-l]$.

5.2 Z Transforms

Table 7: Z transforms

Function	Unilateral Z transform	Region of convergence
$\delta[k]$	1	$z \in \mathbf{C}$
$\delta[k-m]$	z^{-m}	$\lvert z\rvert > 0$, $m = 1, 2, \ldots$
$H[k]$	$\dfrac{z}{z-1}$	$\lvert z\rvert > 1$
k	$\dfrac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
a^k	$\dfrac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
ka^k	$\dfrac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
k^2a^k	$\dfrac{az^2+a^2z}{(z-a)^3}$	$\lvert z\rvert > \lvert a\rvert$
k^3a^k	$\dfrac{az^3+4a^2z^2+a^3z}{(z-a)^4}$	$\lvert z\rvert > \lvert a\rvert$
$(k+1)a^k$	$\dfrac{z^2}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
$\binom{k+m}{m} a^k$	$\dfrac{z^{m+1}}{(z-a)^{m+1}}$	$\lvert z\rvert > \lvert a\rvert$, $m = 2, 3, \ldots$
$\binom{k}{m} a^k$	$\dfrac{a^m z}{(z-a)^{m+1}}$	$\lvert z\rvert > \lvert a\rvert$, $m = 2, 3, \ldots$
$\binom{k+n}{m} a^k$	$\dfrac{a^{m-n}z^{n+1}}{(z-a)^{m+1}}$	$\lvert z\rvert > \lvert a\rvert$, $m = 2, 3, \ldots,$ $n = 1, \ldots, m-1$
$\cos k\alpha$	$\dfrac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1}$	$\lvert z\rvert > 1$
$\sin k\alpha$	$\dfrac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$	$\lvert z\rvert > 1$
$k\cos k\alpha$	$\dfrac{z^3\cos\alpha - 2z^2 + z\cos\alpha}{(z^2 - 2z\cos\alpha + 1)^2}$	$\lvert z\rvert > 1$
$k\sin k\alpha$	$\dfrac{z^3\sin\alpha - z\sin\alpha}{(z^2 - 2z\cos\alpha + 1)^2}$	$\lvert z\rvert > 1$
$\dfrac{a^k}{k!}$	$e^{a/z}$	$\lvert z\rvert > 0$
$\dfrac{1}{k}H[k-1]$	$\ln\dfrac{z}{z-1}$	$\lvert z\rvert > 1$
$\binom{n}{k} a^k b^{n-k}$	$\dfrac{(bz+a)^n}{z^n}$	$\lvert z\rvert > 0$, $n = 1, 2, 3, \ldots$