TATA57/TEN1 2020-06-04 – Solutions

1. (a) Nope. The function has poles at $2 \pm i$, so it is not analytic for $\operatorname{Re} s > 1$.

   (b) Yes. The partial sums are obviously continuous functions so if the convergence is uniform, then the Fourier series converges to something that must be continuous.

   (c) Nope. It is not sufficient since the function u might have a jump at $\pm\pi$. We need to have $u(-\pi) = u(\pi)$, otherwise u is discontinuos (on $\mathbf{R}$) and so the Fourier series can not converge uniformly.

   (d) Nope. The function is discontinuous at ± 1, which is not possible for a Fourier transform of a function from G (which is uniformly continuous).

   (e) Yes. The main theorem from lecture 8 proves that the Fejèr means converges to the mean value of the right- and lefthand limits of the function. This integral is the convolution with the Fejèr kernel, which is one way to write the Fejèr means.

2. Assuming that $y \in X_a$, we take the Laplce transform to find that

$$s^2 Y(s) - sy(0) - y'(0) - 3sY(s) + 3y(0) + 2Y(s) = 10\,\mathcal{L}(\cos 2t)(s-1) = \frac{10(s-1)}{(s-1)^2+4}$$

$$\Leftrightarrow \quad (s^2 - 3s + 2)Y(s) = s - 4 + \frac{10(s-1)}{(s-1)^2+4}.$$

Hence

$$Y(s) = \frac{s-4}{(s-1)(s-2)} + \frac{10}{((s-1)^2+4)(s-2)} = \frac{3}{s-1} - \frac{2}{s-2} - \frac{2s}{(s-1)^2+4} + \frac{2}{s-2}$$
$$= \frac{3}{s-1} - \frac{2(s-1+1)}{(s-1)^2+4} = \frac{3}{s-1} - \frac{2(s-1)}{(s-1)^2+4} - \frac{2}{(s-1)^2+4},$$

so by uniqueness,

$$y(t) = 3e^t - e^t(2\cos 2t + \sin 2t),$$

since this is an exponentially bounded function.

3. We need to find a function y that is at least two times differentiable. Hence y is at least continuous. This implies that y'' is continuous since $y''(x) = 4y(x + \pi/2) + \sin 3x$, where the right-hand side is of (at least) continuous. Thus $y \in C^2$ (at least), which in turn implies that y'' is of class C^2 (at least). Hence $y \in C^3$ (C^4 at least but we could continue this process). Therefore we can write

$$y(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad \Rightarrow \quad y'(x) = \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}, \quad y''(x) = \sum_{k=-\infty}^{\infty} -k^2 c_k e^{ikx}.$$

Hence

$$y''(x) - 4y(x - \pi/2) = \sum_{k=-\infty}^{\infty} (-k^2 - 4e^{-ik\pi/2}) c_k e^{ikx} = \sin 3x = \frac{1}{2i}e^{i3x} - \frac{1}{2i}e^{-i3x}.$$

By uniqueness, we must have

$$(-k^2 - 4e^{-ik\pi/2})c_k = \pm\frac{1}{2i}, \quad k = \pm 3$$

and

$$(-k^2 - 4e^{-ik\pi/2})c_k = 0, \quad k \neq \pm 3.$$

From this follows that

$$c_3 = \frac{1}{2i(-9 - 4e^{-i3\pi/2})} = \frac{4+9i}{2 \cdot 97} \quad \text{and} \quad c_{-3} = \frac{-1}{2i(-9 - 4e^{i3\pi/2})} = \frac{4-9i}{2 \cdot 97}.$$

Moreover, for $k = \pm 2$ we find that

$$-k^2 - 4e^{\mp i\pi} = -4 + 4 = 0,$$

so $c_{\pm 2}$ are arbitrary (complex) constants. For $|k| > 2$, we have $|k^2| > 4$ so $c_k = 0$ is necessary (except for $k = \pm 3$). When $k = 0$ we also see that $c_0 = 0$ is necessary. Therefore

$$\begin{aligned} y(x) &= c_2 e^{i2x} + c_{-2} e^{-i2x} + c_3 e^{i3x} + c_{-3} e^{-i3x} \\ &= A \cos 2x + B \sin 2x + \frac{4}{97}\left(\frac{e^{i3x} + e^{-i3x}}{2}\right) + \frac{9i}{97}\left(\frac{e^{i3x} - e^{-i3x}}{2}\right) \\ &= A \cos 2x + B \sin 2x + \frac{4}{97}\cos 3x - \frac{9}{97}\sin 3x. \end{aligned}$$

4. Clearly $f \in E$. We find that

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi - t)e^{-ikt}\,dt = \cdots = \frac{(-1)^{k+1} i}{k}, \quad k \neq 0,$$

and

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi - t)\,dt = \cdots = \pi.$$

Hence

$$f(t) \sim \pi + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^{k+1} i}{k} e^{ikt}.$$

Furthermore, the function is differentiable for $-\pi < t < \pi$ and has right- and lefthand derivatives at $t = \mp\pi$ (respectively). Hence the Fourier series is convergent and (due to continuity) converges to $f(t)$ for $t \neq (2m+1)\pi$, $m \in \mathbf{Z}$. At these points, we still have the one-sided derivatives, so the Fourier series converges to $(f(x^+) + f(x^-))/2$. In particular,

$$S(\pi) = \frac{f(\pi^-) + f((-\pi)^+)}{2} = \frac{(\pi - \pi) + (\pi - (-\pi))}{2} = \pi$$

and

$$S(-\pi) = \frac{f(\pi^-) + f((-\pi)^+)}{2} = \frac{(\pi - \pi) + (\pi - (-\pi))}{2} = \pi,$$

so $S(k\pi) = \pi$ for $k \in \mathbf{Z}$.

Parseval's identity states that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi - t)^2\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2 = \pi^2 + \sum_{k\neq 0} \frac{1}{k^2} = \pi^2 + 2\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Calculating the integral, we find that

$$\frac{4\pi^2}{3} = \pi^2 + 2\sum_{k=1}^{\infty}\frac{1}{k^2} \quad \Leftrightarrow \quad \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

We can express the Fourier series in real form by writing

$$\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \frac{(-1)^{k+1}i}{k} e^{ikt} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}i}{k}\left(e^{ikt} - e^{-ikt}\right) = -2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\sin kt}{k},$$

so $f(t) = \pi + 2\sum_{k=1}^{\infty} \frac{(-1)^k \sin kt}{k}$ for $t \neq (2m+1)\pi$. If we let $t = \pi/2$, then

$$\frac{\pi}{2} = f(\pi/2) = \pi + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin\left(\frac{k\pi}{2}\right).$$

If $k = 2m$, then $\sin(k\pi/2) = \sin(m\pi) = 0$, and if $k = 2m+1$, then

$$\sin(k\pi/2) = \sin(\pi/2 + m\pi) = (-1)^m.$$

Hence

$$\frac{\pi}{2} = \pi + 2\sum_{m=0}^{\infty} \frac{(-1)^{2m+1}}{2m+1}(-1)^m = \pi - 2\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \quad \Leftrightarrow \quad \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4}.$$

5. Taking the Z transform, we find that

$$zY(z) - zy[0] - Y(z) = \frac{z^3 + z^2}{(z-1)^3}, \quad |z| > 1,$$

since $\mathcal{Z}(k^2) = \mathcal{Z}(k^2 1^k) = (z^2 + z)/(z-1)^3$. Thus

$$Y(z) = \frac{z^3 + z^2}{(z-1)^4} = \frac{1}{z}\frac{z^4}{(z-1)^4} + \frac{1}{z^2}\frac{z^4}{(z-1)^4}, \quad |z| > 1.$$

From a table, we find that $\mathcal{Z}\left(\binom{k+m}{m}\right) = \frac{z^{m+1}}{(z-1)^{m+1}}$ for $|z| > 1$. By uniqueness, we therefore have

$$y[k] = H[k-1]\binom{k+2}{3} + H[k-2]\binom{k+1}{3}.$$

So thus we obtain that $y[0] = 0$ (as expect) and $y[1] = 1$. For $k \geq 2$,

$$\begin{aligned} y[k] &= \binom{k+2}{3} + \binom{k+1}{3} = \frac{(k+2)!}{6\,(k-1)!} + \frac{(k+1)!}{6\,(k-2)!} \\ &= \frac{k(k+1)}{6}(k+2+k-1) = \frac{k(k+1)(2k+1)}{6}. \end{aligned}$$

Note that this formula also holds for $k = 0$ and $k = 1$, so the answer is

$$y[k] = \frac{k(k+1)(2k+1)}{6}, \quad k = 0, 1, 2, \ldots$$

6. We note that the integrand is real and even, so

$$\int_0^\infty \frac{\cos tx}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos tx}{1+x^2}\,dx = \frac{1}{2}\operatorname{Re}\int_{-\infty}^\infty \frac{1}{1+x^2}\,e^{itx}dx.$$

But the integral here is the Fourier transform of $1/(1+x^2)$ at the point $-t$, so we have

$$\frac{1}{2}\operatorname{Re}\mathcal{F}\left(\frac{1}{1+x^2}\right)(-t) = \frac{1}{2}\operatorname{Re}\pi e^{-|-t|} = \frac{\pi}{2}\,e^{-|t|}.$$

For the second integral, we see that

$$\frac{t+\cos x}{t+x+tx^2} = \frac{1+t^{-1}\cos x}{1+x^2+t^{-1}x} \to \frac{1}{1+x^2}, \qquad \text{as } t\to\infty.$$

Now note that

$$\begin{aligned}\left|\frac{1+t^{-1}\cos x}{1+x^2+t^{-1}x} - \frac{1}{1+x^2}\right| &= \left|\frac{(1+t^{-1}\cos x)(1+x^2)-(1+x^2+t^{-1}x)}{(1+x^2)(1+x^2+t^{-1}x)}\right| \\ &= \left|\frac{t^{-1}(1+x^2)\cos x - t^{-1}x}{(1+x^2)(1+x^2+t^{-1}x)}\right| \le \frac{|(1+x^2)\cos x - x|}{t} \le \frac{3}{t} \to 0,\end{aligned}$$

as $t\to\infty$ since $(1+x^2)(1+x^2+t^{-1}x) \ge 1$ for $x\ge 0$. Thus the convergence is uniform and we have

$$\lim_{t\to\infty}\int_0^1 \frac{t+\cos x}{t+x+tx^2}\,dx = \int_0^1 \frac{1}{1+x^2}\,dx = \arctan(1)-\arctan(0) = \frac{\pi}{4}.$$

7. We see that the condition (which is known as a Dini condition) includes something akin to a difference quotient, albeit in some integrated form. Let's try to use the same type of argument we used when proving Dirichlet's theorem (where we knew that the difference quotient was bounded).

   We prove that

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x+t)D_n(t)\,dt = l.$$

   Since $D_n(-t) = D_n(t)$, we see that

$$\int_{-\pi}^{\pi} u(x+t)D_n(t)\,dt = \int_{-\pi}^{\pi} u(x-t)D_n(t)\,dt,$$

   so

$$\int_{-\pi}^{\pi} u(x+t)D_n(t)\,dt = \int_{-\pi}^{\pi} \frac{u(x+t)+u(x-t)}{2}\,D_n(t)\,dt.$$

   Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = 1,$$

   we can write

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{u(x+t)+u(x-t)}{2}\,D_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{u(x+t)+u(x-t)-2l}{2}\,D_n(t)\,dt + l.$$

We need to prove that the integral tends to zero. Recall that $D_n(t) = \dfrac{\sin((2n+1)t/2)}{\sin(t/2)}$, so the integral can be expressed as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{u(x+t)+u(x-t)-2l}{2t}\,\frac{t}{\sin(t/2)}\,\sin((2n+1)t/2)dt.$$

Clearly

$$\frac{u(x+t)+u(x-t)-2l}{2t}\,\frac{t}{\sin(t/2)} \in L^1(-\pi,\pi),$$

so by the Riemann-Lebesgue lemma (for $L^1(-\pi,\pi)$)

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} \frac{u(x+t)+u(x-t)-2l}{2t}\,\frac{t}{\sin(t/2)}\,\sin((2n+1)t/2)dt = 0.$$

This concludes the proof.