LINKÖPINGS UNIVERSITET
Matematiska institutionen
Transform theory

Utbildningskod: TATA57
Modul: TEN1
Datum: 2020-08-20
Institution: MAI

Exam in Transform Theory

TATA57/TEN1 2020-08-20

You are permitted to bring:

- Transformteori - Sammanfattning, Formler & Lexikon (from MAI);
- Table of Formulæ (by Johan Thim);

Since this is a home exam this time, you are **allowed** to use whatever books and sources you want to apart from communicating with other people (speaking, writing, etc). You may use theorems from the course litterature (including the lecture notes). Provide a reference to what theorem you are using in each instance.

The exam is meant to be carried out by yourself without any assistence from other people. You are not permitted to ask questions (during the exam) about the problems on the exam to anyone other than the examiner. Any form of cooperation is disallowed (and will be reported to the disciplinary board if suspected).

Each problem is worth 5 points (for a total of 35 points). No half-points will be awarded in the grading. A solution is deemed good if the question is awarded at least 3 points out of 5. For grade n, you need at least n good answers.

ERASMUS students will have their grades marked according to the scale: A = grade 5, B = grade 4, C = grade 3.

Grading (sufficient limit): 21 points for grade 3. Your solutions need to be complete, well motivated, carefully written and concluded by a clear answer. Be very careful with motivations since these are a huge part of your solutions. Make sure to point out that conditions in theorems you are using hold. Assumptions you make need to be explicit. The exercises are not necessarily in order of difficulty.

Solutions can be found on the homepage a couple of hours after the finished exam.

1. For each of these statements, prove or disprove the claims. You get $+1\,\mathrm{p}$ for each correct answer and $-1\,\mathrm{p}$ for each incorrrect answer. You need to motivate your answers carefully. At worst you will obtain 0 points.

   (a) Since $\mathcal{F}(2/(1+x^2)) = e^{-|\omega|}$, we have $\mathcal{F}(2x/(1+x^2)) = i(d/d\omega)e^{-|w|}$ for $\omega \neq 0$.

   (b) A function $u \in L^2(\mathbf{R})$ also belongs to $L^1(\mathbf{R})$.

   (c) If $U(z) = \mathcal{Z}\,u(z)$, then $u[1] = \lim\limits_{\mathbf{R} \ni z \to \infty} z(U(z) - u[0])$.

   (d) If $S_n(x)$ are the partial sums of the Fourier series for $u \in E$, then $\|S_n - u\|_2 \to 0$ as $n \to \infty$.

   (e) If a series $u(x) = \sum_{k=1}^{\infty} u_k(x)$, where $u_k \in C^1(\mathbf{R})$, converges uniformly, then u is differentiable.

2. Find an absolutely integrable solution to

$$y(t) + \int_{-\infty}^{t} e^{-(t-\tau)} y(\tau)\, d\tau = 3e^{-|t|}, \quad t \in \mathbf{R}.$$

3. Let $f(t) = te^{-it}$ for $-\pi \leq t \leq \pi$. Find the Fourier series for f and show if and to what the Fourier series converges to. Also calculate the series

$$\sum_{k=0}^{\infty} \frac{(-1)^m}{2m+1}.$$

   Also find the Fourier series for f'.

4. Find a solution to

$$y[k+2] - 3y[k+1] + 2y[k] = 2k + 2 \cdot 3^k, \quad k = 0, 1, 2, \ldots,$$

   such that $y[0] = y[1] = 1$.

5. Find all 2π-periodic solutions to $y^{(3)}(x) + 8y(x + \pi/4) = \cos x$.

6. Define $u(t) = \int_0^1 \sqrt{x}\, \sin(tx)\, dx$ for $t \in \mathbf{R}$. Calculate $\int_{-\infty}^{\infty} |u(t)|^2\, dt$.

7. Suppose that $u \in E$ is 2π-periodic and that $\sup\limits_{x \neq y} \dfrac{|u(x) - u(x)|}{|x-y|^{\alpha}} < \infty$ for some $0 < \alpha \leq 1$. Show that there exists a constant C such that the Fourier coefficients $|c_k| \leq C|k|^{-\alpha}$ for $k \neq 0$. Also prove that if $\alpha > 1/2$, then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |S_n(x) - u(x)|^2\, dx \leq Cn^{1-2\alpha}, \quad n = 2, 3, 4, \ldots,$$

   where $S_n(x)$ are the partial sums of the complex Fourier series of $u(x)$.
   Hint: consider the Fourier coefficients for $u(x - \pi/n)$.