# TATA57/TEN1 2020-08-20 – Solutions

1. (a) We can't motivate this "formula" since $x/(1+x^2) \notin L^1(\mathbf{R})$. It might be true for a different Fourier transform, but not for the one we have defined on $G(\mathbf{R})$.

   (b) Nope. For instance the function $x/(1+x^2)$ above does belong to $L^2(\mathbf{R})$ but not $L^1(\mathbf{R})$.

   (c) Yes. Since $z(U(z) - u[0]) = z \sum_{k=1}^{\infty} u[k]z^{-k} = \sum_{k=0}^{\infty} u[k+1]z^{-k}$, we see that if $|z| \to \infty$ then this expression tends to $u[1]$ by the initial value theorem.

   (d) Yes. The ON system used for the Fourier series are closed in $E$ using the inner product from $L^2$.

   (e) No. This is not true in general. We need more information about the series of the derivatives $u_k'$. A famous counter example is Weierstrass' function (continuous but nowhere differentiable).

2. Assuming that $y \in G$, we take the Fourier transform to find that

$$Y(\omega) + \frac{1}{1+i\omega} Y(\omega) = 3\,\mathcal{F}(e^{-|x|}) = \frac{6}{1+\omega^2}$$

since the integral is the convolution of $y(t)$ and $e^{-t}H(t)$. Hence

$$Y(\omega)\left(1 + \frac{1}{1+i\omega}\right) = Y(\omega)\frac{2+i\omega}{1+i\omega} = \frac{6}{1+\omega^2}$$

so

$$\begin{aligned} Y(\omega) &= \frac{6(1+i\omega)}{(1+\omega^2)(2+i\omega)} = \frac{6(1+i\omega)}{(1+i\omega)(1-i\omega)(2+i\omega)} = \frac{6}{(1-i\omega)(2+i\omega)} \\ &= \frac{2}{1-i\omega} + \frac{2}{2+i\omega} \end{aligned}$$

by decomposing into partial fractions. Note that

$$\mathcal{F}(H(-x)e^{x}) = \frac{1}{1-i\omega} \quad \text{and} \quad \mathcal{F}(H(x)e^{-2x}) = \frac{1}{2+i\omega}$$

so by uniqueness,

$$y(x) = 2H(-x)e^{x} + 2H(x)e^{-2x}.$$

This function is absolutely integrable.

3. We consider $f(t)$ periodically extendend from $-\pi \le t < \pi$ to $\mathbf{R}$ (the change at the point $t = \pi$ will not change the Fourier series). Clearly $f \in E$. We find that

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{-it}e^{-ikt}\,dt = \cdots = \frac{(-1)^{k+1}i}{k+1}, \quad k \neq -1,$$

and

$$c_{-1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\,dt = 0.$$

Hence

$$\begin{aligned} f(t) &\sim \sum_{\substack{k=-\infty \\ k\neq -1}}^{\infty} \frac{(-1)^{k+1}i}{k+1} e^{ikt} = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^{k}i}{k} e^{i(k-1)t} = e^{-it}\sum_{k=1}^{\infty} \frac{(-1)^k i}{k}\left(e^{ikt} - e^{-ikt}\right) \\ &= e^{-it}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} 2\sin kt = -2e^{-it}\sum_{k=1}^{\infty}\frac{(-1)^k \sin kt}{k}. \end{aligned}$$

The function $f$ is differentiable for $|t| < \pi$ and has right- and lefthand derivatives at $t = \mp\pi$ (respectively). At $\pm\pi$ there's a jump. The Fourier series is therefore convergent and (due to continuity) converges $f(t)$ for $t \neq (2m+1)\pi$, and

$$S(\pi) = \frac{f(\pi^-) + f((-\pi)^+)}{2} = \frac{\pi e^{i\pi} - \pi e^{-i\pi}}{2} = \frac{-\pi + \pi}{2} = 0.$$

The same happens at every odd multiple of $\pi$, that is, $S((2m+1)\pi) = 0$. This coincides with $f(0) = 0$, so $S(m\pi) = 0$ for all $m \in \mathbf{Z}$. Therefore

$$S(x) = -2e^{-it} \sum_{k=1}^{\infty} \frac{(-1)^k \sin kt}{k} = \begin{cases} te^{-it}, & t \neq m\pi, \\ 0, & t = m\pi. \end{cases}$$

Letting $t = \pi/2$ yields that

$$-\frac{i\pi}{2} = \frac{\pi}{2} e^{-i\pi/2} = f(\pi/2) = -2e^{-i\pi/2} \sum_{k=0}^{\infty} \frac{(-1)^k \sin k\pi/2}{k} = -2i \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1},$$

so

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4}.$$

Since $f(-\pi) \neq f(\pi)$, we can't use $\widehat{f'}[k] = ik\widehat{f}[k]$ (you can verify that this would lead to a divergent series). However,

$$f'(t) = e^{-it} - ite^{-it} = e^{-it} - if(t)$$

so

$$f'(t) \sim e^{-it} + 2ie^{-it} \sum_{\substack{k=-\infty \\ k \neq -1}}^{\infty} \frac{(-1)^{k+1} i}{k+1} e^{ikt} = e^{-it} + \sum_{\substack{k=-\infty \\ k \neq -1}}^{\infty} \frac{2(-1)^k}{k+1} e^{i(k-1)t}.$$

4. Taking the Z transform yields

$$z^2 Y(z) - z^2 - z - 3(zY(z) - z) + 2Y(z) = \frac{2z}{(z-1)^2} + \frac{2z}{z-3} = \frac{2z(z+1)(z-2)}{(z-1)^2(z-3)},$$

if $|z| > 3$, so

$$\begin{aligned} Y(z) &= \frac{z^2 - 2z}{(z-1)(z-2)} + \frac{2z(z+1)}{(z-1)^3(z-3)} \\ &= \frac{z}{z-1} + \frac{3}{z-3} - \frac{3}{z-1} - \frac{2}{(z-1)^3} - \frac{4}{(z-1)^2} \\ &= \frac{z}{z-1} + \frac{1}{z}\left(\frac{3z}{z-3} - \frac{3z}{z-1} - \frac{4z}{(z-1)^2}\right) - \frac{2}{z^3} \cdot \frac{z^3}{(z-1)^3}. \end{aligned}$$

By uniqueness, we use the table to find that

$$\begin{aligned} y[k] &= 1 + H[k-1]\left(3 \cdot 3^{k-1} - 3 - 4(k-1)\right) - 2H[k-3]\binom{k-1}{2} \\ &= 1 + H[k-1]\left(3 \cdot 3^{k-1} - 3 - 4(k-1)\right) - H[k-3](k-1)(k-2). \end{aligned}$$

We see that $y[0] = y[1] = 1$. Furthermore, $y[2] = 3$ and

$$y[k] = 1 + 3^k - 4k + 1 - k^2 + 3k - 2 = 3^k - k^2 - k, \quad k \geq 3.$$

We note that this formula also holds for $k = 0, 1, 2$, so

$$y[k] = 3^k - k^2 - k, \quad k \geq 0.$$

5. We need to find a function $y$ that is at least three times differentiable. Hence $y \in C^2$ at least. This implies that $y^{(3)}$ is of class $C^2$ since $y^{(3)}(x) = -8y(x+\pi/4) + \cos x$, where the right-hand side is of (at least) class $C^2$. Thus $y \in C^5$ (at least). Therefore we can write

$$y(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad \Rightarrow \quad y'(x) = \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}, \quad y''(x) = \sum_{k=-\infty}^{\infty} -k^2 c_k e^{ikx},$$
$$y^{(3)}(x) = \sum_{k=-\infty}^{\infty} -ik^3 c_k e^{ikx}.$$

Hence

$$y^{(3)}(x) + 8y(x+\pi/4) = \sum_{k=-\infty}^{\infty} (-ik^3 + 8e^{ik\pi/4})c_k e^{ikx} = \cos x = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}.$$

By uniqueness, we must have

$$(-ik^3 + 8e^{ik\pi/4})c_k = \frac{1}{2}, \quad k = \pm 1$$

and

$$(-ik^3 + 8e^{ik\pi/4})c_k = 0, \quad k \neq \pm 1.$$

From this follows that

$$c_1 = \frac{1}{2(-i+8e^{i\pi/4})} = \frac{4\sqrt{2} - (4\sqrt{2}-1)i}{2\alpha},$$
$$c_{-1} = \frac{1}{2(i+8e^{-i\pi/4})} = \frac{4\sqrt{2} + (4\sqrt{2}-1)i}{2\alpha},$$

where $\alpha = 32 + (4\sqrt{2}-1)^2$. Moreover, for $k = \pm 2$ we find that

$$-ik^3 + 8e^{ik\pi/4} = 0,$$

so $c_{\pm 2}$ are arbitrary (complex) constants. For $|k| > 2$, we have $|k^3| > 8$ so $c_k = 0$ is necessary. When $k = 0$ we also see that $c_0 = 0$ is necessary. Therefore

$$\begin{aligned} y(x) &= c_1 e^{ix} + c_{-1}e^{-ix} + c_2 e^{i2x} + c_{-2}e^{-i2x} \\ &= \frac{4\sqrt{2}}{2\alpha}\left(e^{ix} + e^{-ix}\right) - i\frac{4\sqrt{2}-1}{2\alpha}\left(e^{ix} - e^{-ix}\right) + A\cos 2x + B\sin 2x \\ &= \frac{4\sqrt{2}}{\alpha}\cos x + \frac{4\sqrt{2}-1}{\alpha}\sin x + A\cos 2x + B\sin 2x. \end{aligned}$$

6. Note that

$$f(t) = \int_0^1 \sqrt{x}\left(\frac{e^{itx} - e^{-itx}}{2i}\right) dx = \frac{1}{2i}\left(\int_0^1 -\sqrt{x}e^{-itx}\, dx + \int_0^1 \sqrt{x}e^{itx}\, dx\right).$$

The second integral requires some massaging, so

$$\int_0^1 \sqrt{x}e^{itx}\, dx = \int_0^1 \sqrt{|x|}e^{itx}\, dx = -\int_0^{-1} \sqrt{|y|}e^{it(-y)}\, dy = \int_{-1}^0 \sqrt{|y|}e^{-ity}\, dy,$$

which means that

$$f(t)=\int_{-1}^{1}\frac{-\operatorname{sgn}(x)\sqrt{|x|}}{2i}e^{-itx}\,dx.$$

In other words, the function $f$ is the Fourier transform of the function

$$g(x)=\frac{i\operatorname{sgn}(x)\sqrt{x}}{2}\big(H(x+1)-H(x-1)\big),\quad x\in\mathbf{R},$$

which is a function from $G(\mathbf{R})$ so $f(t)$ exists as a Fourier transform. Moreover, $g\in L^2(\mathbf{R})$ so Plancherel's theorem is applicable:

$$\int_{-\infty}^{\infty}|f(t)|^2\,dt=2\pi\int_{-\infty}^{\infty}|g(x)|^2\,dx=\frac{\pi}{2}\int_{-1}^{1}|\sqrt{|x|}|^2\,dx=\pi\int_{0}^{1}x\,dx=\frac{\pi}{2}.$$

7. Let $k\neq 0$. Observe that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}u(x-\pi/k)e^{-ikx}\,dx=\cdots=\frac{e^{-ik\pi/k}}{2\pi}\int_{-\pi}^{\pi}u(x)e^{-ikx}\,dx=-c_k,$$

so

$$c_k=\frac{1}{4\pi}\int_{-\pi}^{\pi}\big(u(x)-u(x-\pi/k)\big)e^{-ikx}\,dx.$$

Note now that the condition (which is known as a Hölder condition) implies that there exists a constant $C$ such that

$$|u(x)-u(y)|\le C|x-y|^{\alpha}\quad\text{for every } x,y\in\mathbf{R}.$$

Hence

$$\begin{aligned}\left|\int_{-\pi}^{\pi}\big(u(x)-u(x-\pi/k)\big)e^{-ikx}\,dx\right|&\le\int_{-\pi}^{\pi}\big|u(x)-u(x-\pi/k)\big|\,\big|e^{-ikx}\big|\,dx\\&\le\int_{-\pi}^{\pi}|x-(x-\pi/k)|^{\alpha}\,dx\le 2\pi C\left|\frac{\pi}{k}\right|^{\alpha},\end{aligned}$$

which implies that

$$|c_k|\le\frac{C\pi^{\alpha}}{|k|^{\alpha}},\quad k\neq 0.$$

Now, let $v_n(x)=\sum_{k=-n}^{n}c_ke^{ikx}$ so that $v_n(x)=S_n(X)$ is a partial sum of the Fourier series for $u$. Note that

$$v_n(x)\sim\sum_{k=-n}^{n}c_ke^{ikx}$$

since $v_n(x)$ is defined as its own Fourier series. Consider the Fourier series (by linearity) for the function $u(x)-v_n(x)$:

$$u(x)-v_n(x)\sim\sum_{k=-\infty}^{\infty}c_ke^{ikx}-\sum_{k=-n}^{n}c_ke^{ikx}=\sum_{|k|\ge n+1}c_ke^{ikx}.$$

It now follows from Parseval's identity that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|u(x)-v_n(x)|^2\,dx&=\sum_{|k|\ge n+1}|c_k|^2\le C^2\pi^{2\alpha}\sum_{|k|\ge n+1}\frac{1}{|k|^{2\alpha}}=2C^2\pi^{2\alpha}\sum_{k=n+1}^{\infty}\frac{1}{k^{2\alpha}}\\&\le 2C^2\pi^{2\alpha}\int_{n}^{\infty}\frac{1}{x^{2\alpha}}\,dx=2C^2\pi^{2\alpha}\left[\frac{x^{1-2\alpha}}{1-2\alpha}\right]_n^{\infty}=\frac{2C^2\pi^{2\alpha}}{2\alpha-1}n^{1-2\alpha}.\end{aligned}$$