LINKÖPINGS UNIVERSITET
Matematiska institutionen
Transform theory

Utbildningskod: 93MA54
Modul: STN1
Datum: 2020-10-23
Institution: MAI

Exam in Transform Theory

93MA54/STN1 2020-10-23 8–13

You are permitted to bring:

- Transformteori - Sammanfattning, Formler & Lexikon (from MAI);
- Table of Formulæ (by Johan Thim);

Since this is a home exam this time, you are **allowed** to use whatever books and sources you want to apart from communicating with other people (speaking, writing, etc). You may use theorems from the course literature (including the lecture notes). Provide a reference to what theorem you are using in each instance.

The exam is meant to be carried out by yourself without any assistance from other people. You are not permitted to ask questions (during the exam) about the problems on the exam to anyone other than the examiner. Any form of cooperation is disallowed (and will be reported to the disciplinary board if suspected).

Each problem is worth 5 points (for a total of 35 points). No half-points will be awarded in the grading. A solution is deemed good if the question is awarded at least 3 points out of 5. For grade n, you need at least n good answers.

ERASMUS students will have their grades marked according to the scale: A=grade 5, B=grade 4, C=grade 3.

Grading (sufficient limit): 15 points for grade 3. Your solutions need to be complete, well motivated, carefully written and concluded by a clear answer. Be very careful with motivations since these are a huge part of your solutions. Make sure to point out that conditions in theorems you are using hold. Assumptions you make need to be explicit. The exercises are not necessarily in order of difficulty.

Your solutions need to be **written by hand** unless there are special circumstances. You are allowed to use a sketch board or handheld device with a drawing pen (use **white** background). Do not use a red pen. Enumerate the pages (sorted in the order of the exercises).

Solutions can be found on the homepage a couple of hours after the finished exam.

- Before uploading your solutions in a pdf-file, make sure text and symbols are clearly legible.
- Mark **each** page with course code and **flowID**.

Questions during the exam: see `http://courses.mai.liu.se/GU/TATA57/jour.php`

1. For each of these statements, prove or disprove the claims. You get +1 p for each correct answer and −1 p for each incorrect answer. You need to motivate your answers carefully. At worst you will obtain 0 points.

   (a) If $u \in G(\mathbf{R})$ is real valued, then $|U(-\omega)| = |U(\omega)|$.

   (b) The Fourier transform of an absolutely integrable function can be unbounded.

   (c) For $u, v \in l^1(\mathbf{N})$ we have $\mathcal{Z}(u[k]v[k]) = \mathcal{Z}(u)\,\mathcal{Z}(v)$.

   (d) It is always allowed to integrate an absolutely convergent Fourier series termwise:
   $$\int_a^b \left(\sum_{k=-\infty}^{\infty} c_k e^{ikx} \right) dx = \sum_{k=-\infty}^{\infty} c_k \int_a^b e^{ikx}\, dx, \qquad -\infty < a < b < \infty.$$

   (e) If $u \in l^1(\mathbf{Z})$ and $U(\omega)$ is the discrete time Fourier transform of u, then U has the Fourier coefficients $u[-k]$.

2. Solve the equation $3y[k+2] - 5y[k+1] - 2y[k] = 49 \cdot 2^k$, $k = 0, 1, 2, \ldots$, where $y[0] = 0$ and $y[1] = 49/6$.

3. The function f is given by $f(t) = \pi - t$ for $-\pi \le t \le \pi$. Find the Fourier series for f and show if and to what the Fourier series converges to (for all $t \in \mathbf{R}$). Draw the graph of the Fourier series and calculate the value for the series
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

4. (a) Using the Fourier transform, (formally) find a solution $y(x)$, $x \in \mathbf{R}$, to
   $$y''(x) + y'(x) - 6y(x) = 15e^{-x}H(x),$$
   and verify that $y(0) = -2$ and $y'(0^-) = -2$ (the left-hand limit of y' at $x = 0$).

   (b) Use the unilateral Laplace transform to solve
   $$y''(x) + y'(x) - 6y(x) = 15e^{-x}, \quad x > 0,\ y(0) = y'(0) = -2.$$

   (c) If your answers are different for $x > 0$, why? If your answers are the same for $x > 0$, why?

5. Let $f(x) = |x|$ for $|x| \le 1$ and $f(x) = 0$ for $|x| > 1$. Derive the Fourier transform $F(\omega)$ of $f(x)$. Calculate the limit
$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} F(\omega)\, e^{i\omega t}\, d\omega, \quad t \in \mathbf{R},$$
and show that
$$\int_{-\infty}^{\infty} \left(\frac{\omega \sin\omega + \cos\omega - 1}{\omega^2} \right)^2 d\omega = \frac{\pi}{3}.$$

6. Find a solution to
$$\int_0^t |\sin(t-\tau)| u(\tau)\, d\tau = t^3 + (t-\pi)^3 H(t-\pi), \quad t \ge 0.$$

7. Show that if $u \in G(\mathbf{R})$ satisfies $\displaystyle\int_{-\infty}^{\infty} (1+x^2)^{\alpha/2} |u(x)|\, dx < \infty$ for some $0 < \alpha < 1$, then there exists a constant $C > 0$, independent of u, such that the Fourier transform satisfies
$$|U(\omega) - U(\xi)| \le C|\omega - \xi|^{\alpha} \int_{-\infty}^{\infty} (1+x^2)^{\alpha/2} |u(x)|\, dx, \quad \omega, \xi \in \mathbf{R}.$$