TATA57/TEN1 2020-10-23 – Solutions

1. (a) Yes. This follows since $\overline{u(x)} = u(x)$ and $\overline{e^{i\omega x}} = e^{-i\omega x}$. See the Lecture notes.
   (b) No, this is impossible since $\|\mathcal{F}\, u\|_\infty \leq \|u\|_{L^1}$.
   (c) No, this is not true. There's no easy way to split a series $\displaystyle\sum_{k=0}^{\infty} a_k b_k$.
   (d) Yes. An absolutely convergent Fourier series is uniformly convergent and the terms are continuous functions, so we may exchange the order of summation and integration.
   (e) True. If $u \in l^1(\mathbf{Z})$ then the discrete time Fourier transform is given by

$$U(\omega) = \sum_{k=-\infty}^{\infty} u[k] e^{-i\omega k},$$

   which is a uniformly convergent Fourier series, so

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} u[k] e^{-i\omega k} \right) e^{-i\omega m}\, d\omega = \sum_{k=-\infty}^{\infty} u[k] \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\omega(k+m)}\, d\omega = u[-m],$$

   where we can exchange the order of summation and integration due to uniform convergence and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\omega(k+m)}\, d\omega = 0$$

   if $k + m \neq 0$.

2. Taking the Z transform yields (for $|z| > 2$)

$$3(z^2 Y(z) - z^2 y[0] - z y[1]) - 5(z Y(z) - z y[0]) - 2Y(z) = \frac{49z}{z-2}$$
$$\Leftrightarrow \quad (3z^2 - 5z - 2) Y(z) = \frac{49z}{2} + \frac{49z}{z-2} = \frac{49z^2/2}{z-2},$$

so

$$\begin{aligned} Y(z) &= \frac{49z^2/2}{(z-2)(3z^2-5z-2)} = \frac{14}{(z-2)^2} + \frac{8}{z-2} + \frac{1/6}{z+1/3} \\ &= \frac{7}{z} \frac{2z}{(z-2)^2} + \frac{8}{z} \frac{z}{z-2} + \frac{1}{6z} \frac{z}{z+1/3}. \end{aligned}$$

By uniqueness, we use the table to find that

$$y[k] = 7(k-1) 2^{k-1} H[k-1] + 8 \cdot 2^{k-1} H[k-1] + \frac{1}{6} \left(-\frac{1}{3} \right)^{k-1} H[k-1].$$

We see that $y[0] = 0$ (and $y[1] = 8 + 1/6 = 49/6$). Hence our answer is $y[0] = 0$ and

$$y[k] = (7(k-1) + 8) 2^{k-1} - \frac{1}{2} \cdot \left(-\frac{1}{3} \right)^k = (7k+1) 2^{k-1} - \frac{1}{2} (-3)^{-k}, \quad k \geq 1.$$

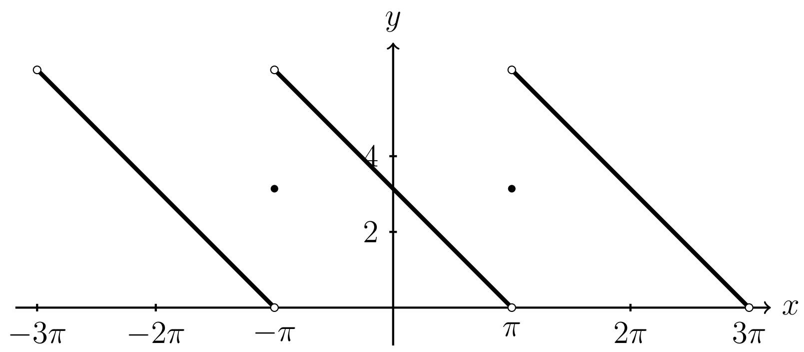
3. Clearly $f \in E$. The function is also differentiable for $-\pi < t < \pi$ and has right- and lefthand derivatives at $t = \mp\pi$ (respectively). Hence the Fourier series is convergent and (due to continuity) converges to $f(t)$ for $t \neq (2m+1)\pi$, $m \in \mathbf{Z}$. At these points, we still have the one-sided derivatives, so the Fourier series converges to $(u(x^+) + u(x^-))/2$. In particular,

$$S(\pi) = \frac{u(\pi^-) + u((-\pi)^+)}{2} = \frac{(\pi - \pi) + (\pi - (-\pi))}{2} = \pi$$

and

$$S(-\pi) = \frac{u(\pi^-) + u((-\pi)^+)}{2} = \frac{(\pi - \pi) + (\pi - (-\pi))}{2} = \pi,$$

so $S((2m+1)\pi) = \pi$ for $m \in \mathbf{Z}$.



To actually find the Fourier series (note that we could answer the question about convergence without finding the coefficients!), integration by parts yields

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi - t)e^{-ikt}\,dt = \cdots = \frac{(-1)^{k+1}i}{k}, \quad k \neq 0,$$

and

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi - t)\,dt = \cdots = \pi.$$

Hence

$$f(t) \sim \pi + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^{k+1}i}{k} e^{ikt}.$$

We can express the Fourier series in real form by writing

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^{k+1}i}{k} e^{ikt} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}i}{k}\left(e^{ikt} - e^{-ikt}\right) = -2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\sin kt}{k},$$

so $f(t) = \pi + 2\sum_{k=1}^{\infty} \frac{(-1)^k \sin kt}{k}$ for $-\pi < t < \pi$ (at least). If we let $t = \pi/2$, then

$$\frac{\pi}{2} = f(\pi/2) = \pi + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin\left(\frac{k\pi}{2}\right).$$

If $k = 2m$, then $\sin(k\pi/2) = \sin(m\pi) = 0$, and if $k = 2m+1$, then

$$\sin(k\pi/2) = \sin(\pi/2 + m\pi) = (-1)^m.$$

Hence

$$\frac{\pi}{2} = \pi + 2\sum_{m=0}^{\infty} \frac{(-1)^{2m+1}}{2m+1}(-1)^m = \pi - 2\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \quad \Leftrightarrow \quad \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4}.$$

4. (a) Assuming that $y \in G$ (and that things makes sense.. which is not a given considering that the right-hand side is discontinuous), we take the Fourier transform to find that

$$(i\omega)^2 Y(\omega) + i\omega Y(\omega) - 6Y(\omega) = \frac{15}{1+i\omega}.$$

Hence

$$Y(\omega) = \frac{15}{((i\omega)^2 + i\omega - 6)(1+i\omega)}.$$

Note that if we let $s = i\omega$ (yeah.. let's save some future work), then

$$\frac{15}{(s^2+s-6)(1+s)} = \frac{15}{(s-2)(s+3)(s+1)} = \frac{-5/2}{s+1} + \frac{1}{s-2} + \frac{3/2}{s+3}.$$

So going back to $s = i\omega$, we find that (by uniqueness)

$$y(x) = -\frac{5}{2}e^{-x}H(x) + \frac{3}{2}e^{-3x}H(x) - e^{2x}H(-x) = \begin{cases} -e^{2x}, & x < 0, \\ -2, & x = 0, \\ (3e^{-3x} - 5e^{-x})/2, & x > 0. \end{cases}$$

The value at $x = 0$ follows (since $H(0) = 1$) from

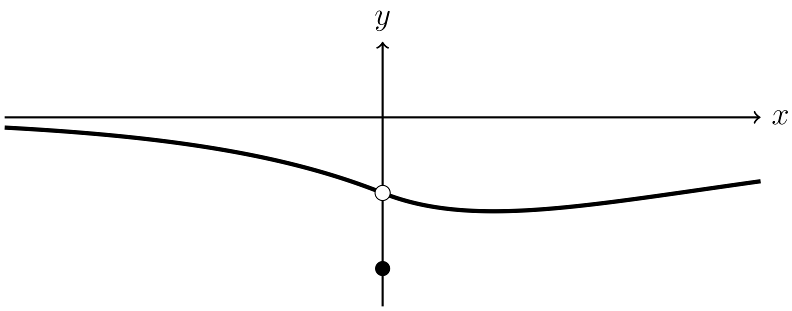
$$y(0) = -\frac{5}{2} + \frac{3}{2} - 1 = -2.$$

This function is clearly absolutely integrable. Moreover, since

$$y(x) = -e^{2x}, \quad x < 0,$$

we have $\lim_{x\to 0^-} y'(x) = -2$.

Notice the difference between $y(0) = -2$ and $\lim_{x\to 0^-} y(x) = -1$. Our function y is discontinuous at $x = 0$. What would happen if we redefined $H(0) = 1/2$ (which is sometimes used)? Is this a coincidence? Would this function be C^2 (so a solution in the classical sense)?



(b) Assuming that y is of exponential order, we find that (with $y(0) = y'(0) = -2$)

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) - 6Y(s) = \frac{15}{1+s}$$
$$\Leftrightarrow \quad (s^2+s-6)Y(s) - (-2s-4) = \frac{15}{1+s},$$

for $\operatorname{Re} s > 3$, so

$$\begin{aligned}Y(s) &= \frac{-5/2}{s+1} + \frac{1}{s-2} + \frac{3/2}{s+3} - \frac{2s+4}{s^2+s-6} \\ &= \frac{-5/2}{s+1} + \frac{1}{s-2} + \frac{3/2}{s+3} - \frac{8/5}{s-2} - \frac{2/5}{s+3} = \frac{-5/2}{s+1} - \frac{3/5}{s-2} + \frac{11/10}{s+3}.\end{aligned}$$

By uniqueness, we find that

$$y(x) = -\frac{5}{2}e^{-x} - \frac{3}{5}e^{2x} + \frac{11}{10}e^{-3x}, \quad x > 0,$$

which is of exponential order.

What would happen with $y(0) = -1$ and $y'(0) = -2$ (the limits in (a))? In this case we would actually find the same solution as in (a) for $x > 0$.

(c) The solutions are different and there are several reasons for this.

- Is the "solution" we found in (a) really a solution to the differential equation? I mean, is this function even differentiable at the origin (see the discussion at the end of part (a))? Would this cause problems for $x > 0$ where everything is smooth, however?
- ...and the answer to that question is yes. With the Fourier transform, we limit ourselves to absolutely integrable functions. Therefore any exponential function e^{ax} with $a \geq 0$ is disqualified.
- The unilateral Laplace transform (as we have defined it) disregards what happens for $t < 0$ and uses right-hand limits to define the values $y(0)$ and $y'(0)$. If we used $y(0^+) = -1$ and $y'(0^+) = -2$ instead, we would actually obtain the same expression as from the Fourier case. Is this obvious?

Side note: solving by regular means (TATA42), we find the same answer as in (b).

5. We observe that $f \in G(\mathbf{R})$ so the Fourier transform exists and

$$\begin{aligned}F(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \int_{-1}^{1} |x|e^{-i\omega x}\,dx = \int_{-1}^{0} -xe^{-i\omega x}\,dx + \int_{0}^{1} xe^{-i\omega x}\,dx \\ &= \int_{0}^{1} te^{i\omega t}\,dt + \int_{0}^{1} xe^{-i\omega x}\,dx = \int_{0}^{1} 2t\cos(\omega t)\,dt = \left[\frac{2t\sin(\omega t)}{\omega}\right]_0^1 - \frac{2}{\omega}\int_0^1 \sin(\omega t)\,dt \\ &= \left[\frac{2t\sin(\omega t)}{\omega} + \frac{2\cos(\omega t)}{\omega^2}\right]_0^1 = \frac{2\omega\sin\omega + 2\cos\omega - 2}{\omega^2}, \quad \omega \neq 0.\end{aligned}$$

At $\omega = 0$, we can either use continuity of F to define $F(0)$ or calculate directly:

$$F(0) = \int_{-\infty}^{\infty} f(x)e^{-i\cdot 0\cdot x}\,dx = \int_{-1}^{1} |x|\,dx = 1.$$

For $x \neq \pm 1$, the function f is differentiable. At $x = \pm 1$, the function f has right- and lefthand derivatives. Hence by Dirichlet's theorem, we find that

$$\lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} F(\omega)e^{i\omega x}\,d\omega = \begin{cases} |x|, & |x| < 1, \\ (0+1)/2 = 1/2, & x = \pm 1, \\ 0, & |x| > 1. \end{cases}$$

To calculate the integral, notice that the integrand is $|F(\omega)/2|^2$, which is the square of the Fourier transform of $f(x)/2$. Hence, since $f \in G(\mathbf{R}) \cap L^2(\mathbf{R})$, we find that Plancherel's theorem implies the desired result:

$$\int_{-\infty}^{\infty} \left(\frac{\omega \sin\omega + \cos\omega - 1}{\omega^2}\right)^2 dt = 2\pi \int_{-\infty}^{\infty} (f(x)/2)^2\, dx = \frac{\pi}{2} \int_{-1}^{1} x^2\, dx = \frac{\pi}{2} \cdot \frac{2}{3} = \frac{\pi}{3}.$$

6. The left-hand side is the convolution of $f(t) = |\sin t|$ with $u(t)$. We need the Laplace transform of $f(t)$. Note that this is a π-periodic function, so

$$\begin{aligned} F(s) &= \frac{1}{1-e^{-\pi s}} \int_0^{\pi} |\sin t| e^{-st}\, dt = \frac{1}{1-e^{-\pi s}} \int_0^{\pi} \sin t e^{-st}\, dt \\ &= \cdots = \frac{1}{1-e^{-\pi s}} \cdot \frac{1+e^{-\pi s}}{1+s^2}, \quad \operatorname{Re} s > 0. \end{aligned}$$

Moreover,

$$\mathcal{L}(t^3 + (t-\pi)^3 H(t-\pi)) = \frac{3!}{s^4}\left(1+e^{-\pi s}\right), \quad \operatorname{Re} s > 0.$$

Thus we can take the Laplace transform of the equation and obtain that

$$\frac{1}{1-e^{-\pi s}} \cdot \frac{1+e^{-\pi s}}{1+s^2} U(s) = \frac{3!}{s^4}\left(1+e^{-\pi s}\right) \quad \Leftrightarrow \quad U(s) = \left(1-e^{-\pi s}\right)\frac{6(1+s^2)}{s^4}.$$

We see that

$$\mathcal{L}(t^3 + 6t) = \frac{6}{s^4} + \frac{6}{s^2},$$

so by uniqueness,

$$u(t) = t^3 + 6t - H(t-\pi)\left((t-\pi)^3 + 6(t-\pi)\right), \quad t \geq 0.$$

7. Following the idea behind the proof of uniform continuity for the Fourier transform of a function in G, we write

$$\begin{aligned} |U(\omega) - U(\xi)| &= \left| \int_{-\infty}^{\infty} u(x)\left(e^{-i\omega x} - e^{-i\xi x}\right) dx \right| \leq \int_{-\infty}^{\infty} |u(x)| \left|e^{-i\omega x} - e^{-i\xi x}\right| dx \\ &= \int_{-\infty}^{\infty} (1+x^2)^{\alpha/2} |u(x)| \cdot \frac{\left|e^{-i\omega x} - e^{-i\xi x}\right|}{(1+x^2)^{\alpha/2}}\, dx. \end{aligned}$$

Recall now that

$$\left|e^{-i\omega x} - e^{-i\xi x}\right| \leq \left|e^{-i\omega x}\right| + \left|e^{-i\xi x}\right| = 2$$

and for $\alpha, \beta \in \mathbf{R}$ we have

$$\begin{aligned} \left|e^{i\alpha} - e^{i\beta}\right|^2 &= |\cos\alpha + i\sin\alpha - \cos\beta - i\sin\beta|^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 \\ &= \cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta - 2\left(\cos\alpha\cos\beta + \sin\alpha\sin\beta\right) \\ &= 2(1-\cos(\alpha-\beta)) = 4\sin^2\left(\frac{\alpha-\beta}{2}\right) \leq 4\left(\frac{\alpha-\beta}{2}\right)^2 = (\alpha-\beta)^2, \end{aligned}$$

since $|\sin x| \leq |x|$ for $x \in \mathbf{R}$. This implies that

$$\left|e^{-i\omega x} - e^{-i\xi x}\right| \leq |\omega - \xi||x|.$$

Hence

$$\frac{\left|e^{-i\omega x}-e^{-i\xi x}\right|}{(1+x^2)^{\alpha/2}} \le \frac{2}{(1+x^2)^{\alpha/2}} \qquad \text{and} \qquad \frac{\left|e^{-i\omega x}-e^{-i\xi x}\right|}{(1+x^2)^{\alpha/2}} \le \frac{|\omega-\xi||x|}{(1+x^2)^{\alpha/2}}.$$

Noting that the two expressions in the right-hand sides are equal if $|\omega-\xi||x|=2$, we see that

$$\begin{aligned} |\omega-\xi||x|>2 \quad \Rightarrow \quad \frac{\left|e^{-i\omega x}-e^{-i\xi x}\right|}{(1+x^2)^{\alpha/2}} &\le \frac{2}{(1+x^2)^{\alpha/2}} \le \frac{2}{(1+(2/|\omega-\xi|)^2)^{\alpha/2}} \\ &\le \frac{2}{(4/|\omega-\xi|^2)^{\alpha/2}} = 2^{1-\alpha}|\omega-\xi|^{\alpha} \end{aligned}$$

and

$$|\omega-\xi||x|\le 2 \quad \Rightarrow \quad \frac{\left|e^{-i\omega x}-e^{-i\xi x}\right|}{(1+x^2)^{\alpha/2}} \le \frac{|\omega-\xi||x|}{(1+x^2)^{\alpha/2}}.$$

The function $t\mapsto t(1+t^2)^{\alpha/2}$ is increasing for $t>0$, so we obtain the same upper bound in the second case as in the previous case:

$$\frac{|\omega-\xi||x|}{(1+x^2)^{\alpha/2}} \le \frac{2}{(1+(2/|\omega-\xi|)^2)^{\alpha/2}} \le 2^{1-\alpha}|\omega-\xi|^{\alpha}.$$

Hence

$$\int_{-\infty}^{\infty}(1+x^2)^{\alpha/2}|u(x)|\cdot\frac{\left|e^{-i\omega x}-e^{-i\xi x}\right|}{(1+x^2)^{\alpha/2}}\,dx \le 2^{1-\alpha}|\omega-\xi|^{\alpha}\int_{-\infty}^{\infty}(1+x^2)^{\alpha/2}|u(x)|\,dx.$$