LINKÖPINGS UNIVERSITET
Matematiska institutionen
Transform theory

Utbildningskod: TATA57
Modul: TEN1
Datum: 2020-xx-xx
Institution: MAI

(example) Exam in Transform Theory

TATA57/TEN1 2020-xx-xx

You are permitted to use:

- Transformteori - Sammanfattning, Formler & Lexikon (from MAI);
- Table of Formulæ (by Johan Thim);

Since this is a home exam this time, you are **allowed** to use whatever books and sources you want to apart from communicating with other people (speaking, writing, etc). You may use theorems from the course literature (including the lecture notes). Provide a reference to what theorem you are using in each instance.

The exam is meant to be carried out by yourself without any assistance from other people. You are not permitted to ask questions (during the exam) about the problems on the exam to anyone other than the examiner. Any form of cooperation is disallowed (and will be reported to the disciplinary board if suspected).

Each problem is worth 5 points (for a total of 35 points). No half-points will be awarded in the grading. A solution is deemed good if the question is awarded at least 3 points out of 5. For grade n, you need at least n good answers.

ERASMUS students will have their grades marked according to the scale: A = grade 5, B = grade 4, C = grade 3.

Grading (sufficient limit): 21 points for grade 3. Your solutions need to be complete, well motivated, carefully written and concluded by a clear answer. Be very careful with motivations since these are a huge part of your solutions. Make sure to point out that conditions in theorems you are using hold. Assumptions you make need to be explicit. The exercises are not necessarily in order of difficulty.

Your solutions need to be **written by hand** unless there are special circumstances. You are allowed to use a sketch board or handheld device with a drawing pen (use **white** background). Do not use a red pen. Enumerate the pages (sorted in the order of the exercises).

Solutions can be found on the homepage a couple of hours after the finished exam.

- Before uploading your solutions in a pdf-file, make sure text and symbols are clearly legible.
- Mark **each** page with course code and **flowID**.

Questions during the exam: see `http://courses.mai.liu.se/GU/TATA57/jour.php`

1. For each of these statements, prove or disprove the claims. You get +1 p for each correct answer and −1 p for each incorrrect answer. You need to motivate your answers carefully. At worst you will obtain 0 points.

   (a) The function $f(t) = |x|^{1/3}$, $0 < |t| \le \pi$ and $f(0) = 1$ has a Fourier series.

   (b) Let u (of exponential order) have the Laplace transform $\dfrac{1}{s^4 + s + 1}$. Since $sU(s) \to 0$ as $\mathbf{R} \ni s \to 0^+$, we have $\lim_{t\to\infty} u(t) = 0$.

   (c) An absolutely convergent Fourier series is uniformly convergent.

   (d) The function $U(z) = 1 + z^2$ is the Z transform of some $u \in l^1(\mathbf{N})$.

   (e) The function $\left(\dfrac{\sin t}{t}\right)^2$ belongs to $G(\mathbf{R})$.

2. Find an absolutely integrable solution to

$$y'(t) + 2y(t) = 4e^{-2|t|}, \quad t \in \mathbf{R},$$

   and show that there's no non-zero 2π-periodic solution to

$$y'(t) + 2y(t + \pi/3) = 0, \quad t \in \mathbf{R}.$$

3. The function f is even with period 2π and is given by $f(t) = \pi - t$ for $0 \le t \le \pi$. Find the Fourierseries for f and show if and to what the Fourierseries converges to. Also calculate the values for the series

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \qquad \text{and} \qquad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}.$$

4. Find a solution to

$$y[k] + \sum_{m=0}^{k} 2^{k-m} y[m] = 8k\, 3^{k-1}, \quad k = 0, 1, 2, \ldots.$$

5. Let $f(x) = \cos x$ for $|x| < \pi$ and $f(x) = 0$ if $|x| \ge \pi$.

   (a) Find the Fouriertransform $F(\omega)$ of f. (3p)

   (b) Find the limit $\displaystyle\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} F(\omega) e^{i\omega x}\, dx$ for every $x \in \mathbf{R}$. (1p)

   (c) Calculate $\displaystyle\int_{-\infty}^{\infty} \left(\frac{2\omega \sin \pi\omega}{1 - \omega^2}\right)^2 d\omega.$ (1p)

6. Let u be of exponential growth, that is $|u(t)| \le Ce^{at}$ for some $C > 0$ and $a > 0$. For which integers $n \ge 0$ does the integral equation

$$\int_0^t u(t - \tau) \sin \tau \, d\tau = t^n, \quad t \ge 0,$$

   have a solution u of exponential order on $[0, \infty[$? Find the solutions in these cases.

7. Show that $u(x) = \displaystyle\sum_{k=0}^{\infty} \int_0^x \frac{t^k\, dt}{k^3 + t^{2k}}$ is convergent for every $x \in \mathbf{R}$ and that $u \in C^1(\mathbf{R})$.