TATA57/TEN1 2020 example exam – Solutions

1. (a) Yes. The function is continuous for $-\pi \le x < 0$ and $0 < x \le \pi$ with onesided limits at 0, so therefore it belongs to E. Thus it has a Fourier series.
   (b) No. This is not clear. We do not know that $u(t)$ has a limit as $t \to \infty$ (this is assumed to use the final value theorem).
   (c) Yes. An absolutely convergent Fourier series will be uniformly convergent. This follows by the Weierstrass M-test (how?).
   (d) Nope. The Z transform has to have a limit as $|z| \to \infty$.
   (e) Yes. The function is continuous and $|\cdot| \le 1/t^2$ for large t, which is sufficient for absolute integrability.

2. Assuming that $y \in G$, we take the Fourier transform to find that

$$i\omega Y(\omega) + 2Y(\omega) = 4\,\mathcal{F}(e^{-2|x|}) = \frac{16}{4+\omega^2}.$$

Hence

$$Y(\omega) = \frac{16}{(4+\omega^2)(2+i\omega)} = \frac{4}{(2+i\omega)^2} + \frac{1}{2+i\omega} + \frac{1}{2-i\omega} = \frac{4}{(2+i\omega)^2} + \frac{4}{4+\omega^2}.$$

Note that

$$\frac{4}{(2+i\omega)^2} = 4i\frac{d}{d\omega}\left(\frac{1}{2+i\omega}\right) = \mathcal{F}\big(4xe^{-2x}H(x)\big)$$

so by uniqueness,

$$y(x) = 4xe^{-2x}H(x) + e^{-2|x|}.$$

This function is absolutely integrable.

Since y needs to be differentiable, we know that y is continuous. The fact that

$$y'(t) = -2y(t+\pi/3)$$

then implies that y' is continuous. Hence y is $C^1(\mathbf{R})$. But then $y' \in C^1(\mathbf{R})$ as well, which means that $y \in C^2(\mathbf{R})$. This continues ad nauseum, so y must be a very smooth function. For our purposes, $y \in C^2(\mathbf{R})$ is sufficient for writing that

$$y(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \quad \text{and} \quad y'(t) = \sum_{k=-\infty}^{\infty} ikc_k e^{ikt}.$$

Why? Well if $y \in C^2(\mathbf{R})$, then both y and y' have uniformly convergent Fourier series and the Fourier series of y' can be found through termwise differentiation of the Fourier series for y. Hence

$$0 = y'(t) + 2y(t+\pi/3) = \sum_{k=-\infty}^{\infty} \left(ik + 2e^{ik\pi/3}\right) c_k e^{ikt}.$$

Hence $ik + 2e^{ik\pi/3} = 0$ or $c_k = 0$. We note that for $k = 0$, we have $i \cdot 0 + 2 \ne 0$ so $c_0 = 0$. If $k = \pm 1$, then

$$i(\pm 1) + 2e^{i(\pm 1)\pi/3} = i(\pm 1) + 2\left(\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right) \ne 0$$

so $c_{\pm 1} = 0$. Similarly,

$$i(\pm 2) + 2e^{i(\pm 2)\pi/3} = i(\pm 2) + 2\left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right) \neq 0$$

so $c_{\pm 2} = 0$ is necessary. For $|k| > 2$, it is impossible to find a solution to $ik + 2e^{ik\pi/3}$ since $|2e^{ik\pi/3}| = 2$. Hence $c_k = 0$ if $|k| > 2$. Therefore, only $y(t) = 0$ is 2π-periodic and solves the equation.

3. The even extension of f to $[-\pi, \pi]$ is given by $f(t) = \pi - |t|$. We consider $f(t)$ periodically extended to $\mathbf{R}$. Clearly $f \in E$. We find that

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi - |t|)e^{-ikt}\, dt = \frac{1}{2\pi}\left(\left[\frac{(\pi - |t|)}{-ik}\, e^{-ikt}\right]_{-\pi}^{\pi} + \frac{1}{ik}\int_{-\pi}^{\pi} -\operatorname{sgn}(t)e^{-ikt}\, dt\right) \\ &= \frac{1}{i2\pi k}\int_{-\pi}^{0} e^{-ikt}\, dt - \frac{1}{i2\pi k}\int_{0}^{\pi} e^{-ikt}\, dt = \left[\frac{1}{2k^2\pi}\, e^{-ikt}\right]_{-\pi}^{0} - \left[\frac{1}{2k^2\pi}\, e^{-ikt}\right]_{0}^{\pi} \\ &= \frac{1}{i2\pi k^2}\left(1 - e^{ik\pi} - e^{-ik\pi} + 1\right) = \frac{1 - (-1)^k}{\pi k^2}, \quad k \neq 0, \end{aligned}$$

and

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi - |t|)\, dt = \cdots = \frac{\pi}{2}.$$

Hence

$$\begin{aligned} f(t) &\sim \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1 - (-1)^k}{\pi k^2} e^{ikt} = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{1 - (-1)^k}{\pi k^2}\left(e^{ikt} + e^{-ikt}\right) \\ &= \frac{\pi}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{1 - (-1)^k}{k^2} 2\cos kt = \frac{\pi}{2} + \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\cos(2m+1)t}{(2m+1)^2}. \end{aligned}$$

Furthermore, the function is differentiable for $0 < |t| < \pi$ and has right- and lefthand derivatives at $t = 0$ and at $t = \mp\pi$ (respectively). Moreover, the periodic extension of f is continuous at every point. Hence the Fourier series is convergent and (due to continuity) converges $f(t)$, so

$$f(t) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\cos(2m+1)t}{(2m+1)^2}, \quad t \in \mathbf{R}.$$

Letting $t = 0$ yields that

$$\pi = f(0) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \quad \Rightarrow \quad \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$

Moreover, we find that

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |c_k|^2 &= \frac{\pi^2}{4} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left|\frac{1 - (-1)^k}{\pi k^2}\right|^2 = \frac{\pi^2}{4} + 2\sum_{k=1}^{\infty} \frac{(1 - (-1)^k)^2}{\pi^2 k^4} \\ &= \frac{\pi^2}{4} + \frac{8}{\pi^2}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^4}, \end{aligned}$$

so by Parseval's identity,

$$\begin{aligned}\frac{\pi^2}{4}+\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^4}&=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|^2\,dt=\frac{1}{\pi}\int_0^{\pi}(\pi-t)^2\,dt=\frac{\pi^2}{3}\\ \Leftrightarrow\quad\sum_{m=0}^{\infty}\frac{1}{(2m+1)^4}&=\frac{\pi^4}{8\cdot 12}=\frac{\pi^4}{96}.\end{aligned}$$

4. Taking the Z transform, we find that

$$\begin{aligned}Y(z)+\mathcal{Z}(2^k)Y(z)=\frac{8}{3}\,\mathcal{Z}(k3^k)\quad&\Leftrightarrow\quad Y(z)+\frac{z}{z-2}Y(z)=-\frac{8}{3}z\frac{d}{dz}\frac{z}{z-3}=\frac{8z}{(z-3)^2}\\ &\Leftrightarrow\quad Y(z)\frac{2z-2}{z-2}=\frac{8z}{(z-3)^2}\end{aligned}$$

for $\operatorname{Re} z>3$. Hence

$$\begin{aligned}Y(z)&=\frac{4z(z-2)}{(z-3)^2(z-1)}=\frac{6}{(z-3)^2}+\frac{5}{z-3}-\frac{1}{z-1}\\ &=\frac{1}{z}\left(\frac{6z}{(z-3)^2}+\frac{5z}{z-3}-\frac{z}{z-1}\right).\end{aligned}$$

By uniqueness, we find that

$$\begin{aligned}y[k]&=H[k-1]\left(2(k-1)3^{k-1}+5\cdot 3^{k-1}-1\right)=\begin{cases}0, & k=0,\\ (2k+3)3^{k-1}-1, & k=1,2,3,\dots\end{cases}\\ &=(2k+3)3^{k-1}-1,\quad k=0,1,2,\dots\end{aligned}$$

5. Since $f\in G(\mathbf{R})$ (continuous function with compact support), we know that the Fourier transform $F(\omega)$ exists for all ω. If $\omega\neq\pm1$, we find that

$$\begin{aligned}F(\omega)&=\int_{-\pi}^{\pi}\cos x e^{-i\omega x}\,dt=\frac{1}{2}\int_{-\pi}^{\pi}\left(e^{-i(\omega-1)x}+e^{-i(\omega+1)x}\right)\,dx\\ &=\frac{1}{2}\left[\frac{e^{-i(\omega-1)x}}{-i(\omega-1)}+\frac{e^{-i(\omega+1)x}}{-i(\omega+1)}\right]_{-\pi}^{\pi}\\ &=\frac{1}{2}\left(\frac{e^{-i(\omega-1)\pi}}{-i(\omega-1)}+\frac{e^{-i(\omega+1)\pi}}{-i(\omega+1)}-\frac{e^{i(\omega-1)\pi}}{-i(\omega-1)}-\frac{e^{i(\omega+1)\pi}}{-i(\omega+1)}\right)\\ &=\frac{1}{2}\left(\frac{e^{-i\omega\pi}}{i(\omega-1)}+\frac{e^{-i\omega\pi}}{i(\omega+1)}-\frac{e^{i\omega\pi}}{i(\omega-1)}-\frac{e^{i\omega\pi}}{i(\omega+1)}\right)\\ &=-\frac{\sin\pi\omega}{\omega-1}-\frac{\sin\pi\omega}{\omega+1}=\frac{2\omega\sin\pi\omega}{1-\omega^2}.\end{aligned}$$

We can find $F(\omega)$ by continuity (since $f\in G$ we know that F is continuous):

$$\lim_{\omega\to\pm1}F(\omega)=\pi,$$

so $F(\pm1)=\pi$. Alternatively, we can put $\omega=\pm1$ in the integral above and redo the calculation.

By Dirichlet's theorem, we know that if $f \in G(\mathbf{R})$ has one-sided derivatives at x, then

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} F(\omega) e^{i\omega x}\, d\omega = \frac{f(x^+) + f(x^-)}{2} = \begin{cases} \cos x, & |x| < \pi, \\ -1/2, & x = \pm\pi, \\ 0, & |x| > \pi, \end{cases}$$

since f has onesided derivatives at all points.

Since $f \in L^2(\mathbf{R}) \cap G(\mathbf{R})$, Plancherel's formula implies that

$$\int_{-\infty}^{\infty} \left(\frac{2\omega \sin \pi\omega}{1-\omega^2} \right)^2 d\omega = 2\pi \int_{-\pi}^{\pi} |\cos x|^2\, dx = 2\pi \int_{-\pi}^{\pi} \frac{1+\cos 2x}{2}\, dx = 2\pi^2.$$

6. The left-hand side is a convolution of u with sin, so taking the Laplace transform (assuming that $u \in X_a$) shows that

$$U(s)\frac{1}{1+s^2} = \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \qquad \operatorname{Re} s > 0.$$

So if $\operatorname{Re} s > 0$, we find that

$$U(s) = \frac{(1+s^2)n!}{s^{n+1}} = \frac{n!}{s^{n+1}} + \frac{n!}{s^{n-1}}.$$

Clearly it is impossible that $n = 0$ or $n = 1$ since this would mean that $U(s)$ doesn't have the limit zero as $\mathbf{R} \ni s \to \infty$, so we have to (at least) assume that $n \geq 2$. For $n \geq 2$, we find that

$$U(s) = \frac{n!}{s^{n+1}} + n(n-1) \cdot \frac{(n-2)!}{s^{n-1}},$$

so by uniqueness we have $u(t) = t^n + n(n-1)t^{n-2}$, $t \geq 0$.

7. First we show that $u(x)$ exists for $x \in \mathbf{R}$. Let

$$v(t) = \sum_{k=0}^{\infty} \frac{t^k}{k^3 + t^{2k}} = \sum_{k=0}^{\infty} v_k(t), \qquad t \in \mathbf{R}.$$

We prove that this series is uniformly convergent. To do this, we will show that there exists a sequence M_k such that $|v_k(t)| \leq M_k$. Notice that

$$v_k'(t) = \frac{kt^{k-1}\big(k^3 - t^{2k}\big)}{(k^3 + t^{2k})^2},$$

so

$$v_k'(t) = 0 \quad \Leftrightarrow \quad k^3 = t^{2k} \quad \Leftrightarrow \quad t = k^{3/(2k)}$$

and $v_k'(t)$ has the sign change $+\ 0\ -$ around the point $t = k^{3/(2k)}$, so it is indeed the maximum we're looking for. Hence

$$|v_k(t)| \leq v_k(k^{3/(2k)}) = \frac{k^{3/2}}{k^3 + k^3} = \frac{1}{2}k^{-3/2}$$

and so by the Weierstrass M-test, we know that $\sum_{k=0}^{\infty} v_k(t)$ is uniformly convergent for $t \in \mathbf{R}$ since $\sum_{k=1}^{\infty} k^{-3/2} < \infty$. Hence $v(t)$ is continuous (since every v_k is continuous) and due to the uniform convergence, we may change the order of summation and integration:

$$u(x) = \sum_{k=0}^{\infty} \int_0^x v_k(t)\, dt = \int_0^x \left(\sum_{k=0}^{\infty} v_k(t) \right) dt = \int_0^x v(t)\, dt.$$

Since v is continuous, it is clear that $u'(x) = v(x)$ for $x \in \mathbf{R}$ (by the fundamental theorem of calculus), so clearly $u \in C^1(\mathbf{R})$.