Transform theory 2021-06-03 – Solutions

1. (a) No. The function is neither bounded nor continuous (at $\omega = -1$), so it cannot be the Fourier transform of an integrable function.

   (b) No. This is impossible since the Z transform has to have a limit as $|z| \to \infty$ (and this limit is equal to $u[0]$).

   (c) Yes. The function is differentiable on $]-\pi, \pi[$, has suitable one-sided derivatives at $\pm\pi$, is continuous on $[-\pi, \pi]$, and $u(\pi) = u(-\pi)$. Therefore the convergence of the Fourier series is uniform on $[-\pi, \pi]$ (and therefore on $\mathbf{R}$).

   (d) No, uniform convergence would imply convergence of the integrals. But not necessarily the other way around. For instance, consider $u_k(x) = x^k$.

   (e) Not true. From the table, we can for instance find that $\mathcal{F}(e^{-|x|}) = 2/(1+\omega^2)$, but $e^{-|x|} \leq 1$ and $\displaystyle\int_{-\infty}^{\infty} \frac{2}{1+\omega^2}\, d\omega = 2\pi > 1$.

   Answer: No, No, Yes, No, No.

2. Clearly $f \in E$. We find that

$$\begin{aligned}
c_k &= \frac{1}{2\pi}\int_0^{\pi} (\pi - t)e^{-ikt}\, dt = \frac{1}{2\pi}\left[\frac{(\pi - t)\, e^{-ikt}}{-ik}\right]_0^{\pi} - \frac{1}{2\pi ik}\int_0^{\pi} e^{-ikt}\, dt \\
&= \frac{\pi}{2\pi ik} + \frac{1}{2\pi i^2 k^2}\left[e^{-ikt}\right]_0^{\pi} = \frac{\pi}{2\pi ik} + \frac{(-1)^k - 1}{2\pi i^2 k^2} = \frac{-ik\pi}{2\pi k^2} + \frac{1 - (-1)^k}{2\pi k^2} \\
&= \frac{1 - (-1)^k - ik\pi}{2\pi k^2}, \quad k \neq 0,
\end{aligned}$$

and

$$c_0 = \frac{1}{2\pi}\int_0^{\pi} (\pi - t)\, dt = \frac{\pi}{4}.$$

Hence

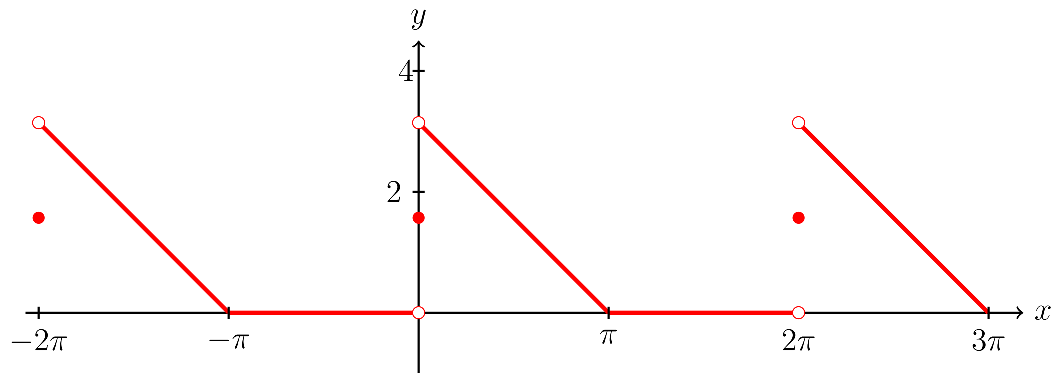
$$f(t) \sim \frac{\pi}{4} + \frac{1}{2\pi}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1 - (-1)^k - ik\pi}{k^2}\, e^{ikt}.$$

Furthermore, the function is differentiable for $-\pi < t < 0$ and $0 < t < \pi$. The function has right- and lefthand derivatives at $t = \mp\pi$ (respectively) and at $t = 0$. Hence – by Dirichlet's theorem – the Fourier series is convergent and converges to $f(t)$ (due to continuity) for $-\pi < t < 0$ and $0 < t < \pi$. We note that the periodic extension of $f(t)$ is continuous also at $t = (2k+1)\pi$, $k \in \mathbf{Z}$, so the Fourier series converges to $f((2k+1)\pi) = 0$ at these points. At even multiples of π, we observe that, e.g.,

$$S(0) = \frac{u(0^-) + u(0^+)}{2} = \frac{0 + \pi}{2} = \frac{\pi}{2}$$

and this repeats so that $S(2k\pi) = \dfrac{\pi}{2}$ for $k \in \mathbf{Z}$.

With this information, we can draw the graph of the Fourier series.



Answer: $f(t) \sim \dfrac{\pi}{4} + \dfrac{1}{2\pi} \displaystyle\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \dfrac{1-(-1)^k - ik\pi}{k^2}\, e^{ikt}$; see above.

3. (a) Let $g(t) = \cos 3t$. Then $G(s) = \dfrac{s}{s^2+3^2} = \dfrac{s}{s^2+9}$, $\operatorname{Re} s > 0$. By the "s-shift" property,

$$\mathcal{L}(e^{-5t}f(t)) = G(s-(-5)) = G(s+5), \quad \operatorname{Re} s > -5,$$

and by "time-multiplication,"

$$\mathcal{L}(te^{-5t}f(t)) = -\frac{d}{ds}\left(G(s+5)\right) = -F'(s+5) = -\frac{(s+5)^2+9-2(s+5)^2}{((s+5)^2+9)^2}$$
$$= \frac{(s+5)^2-9}{((s+5)^2+9)^2}, \quad \operatorname{Re} s > -5.$$

(b) Noting that this integral is the Laplace transform of $g(t)$ evaluated at $s=0$ (which is OK since $0 > -5$), we obtain

$$\int_0^\infty te^{-5t}\cos 3t\,dt = \mathcal{L}(te^{-5t}\cos 3t)(0) = \left.\frac{(s+5)^2-9}{((s+5)^2+9)^2}\right|_{s=0}$$
$$= \frac{25-9}{(25+9)^2} = \frac{16}{1156} = \frac{4}{289}.$$

Answer: (a) $\dfrac{(s+5)^2-9}{(s+5)^2+9)^2}$, $\operatorname{Re} s > -5$ (b) $\dfrac{4}{289}$.

4. Taking the Z transform yields

$$z^2U(z) - 5z^2 - 5z - (zU(z) - 5z) - 6U(z) = \frac{25z}{z-3} \quad \Leftrightarrow \quad \left(z^2 - z - 6\right)U(z) = 5z^2 + \frac{25z}{z-3},$$

if $|z| > 3$. Moreover, $z^2 - z - 6 = (z-3)(z+2)$, so

$$U(z) = \frac{5z^2(z-3)+25z}{(z+2)(z-3)^2} = z\cdot\frac{5z^2-15z+25}{(z+2)(z-3)^2}, \quad \operatorname{Re} z > 3.$$

We decompose into partial fractions:

$$z\cdot\frac{5z^2-15z+25}{(z+2)(z-3)^2} = z\left(\frac{5}{(z-3)^2} + \frac{2}{z-3} + \frac{3}{z+2}\right) = \frac{5z}{(z-3)^2} + \frac{2z}{z-3} + \frac{3z}{z+2}.$$

By uniqueness, we use the table to find that

$$u[k] = 5k3^{k-1} + 2 \cdot 3^k + 3 \cdot (-2)^k.$$

We see that $y[0] = y[1] = 5$ and can directly verify that this solves the equation.

Answer: $u[k] = 5k3^{k-1} + 2 \cdot 3^k + 3 \cdot (-2)^k$, $k = 0, 1, 2, \ldots$

5. Assuming that $y, y', y'' \in G$, we take the Fourier transform to find that

$$(i\omega)^2 Y(\omega) + 3i\omega Y(\omega) + 2Y(\omega) = \mathcal{F}(te^{-x}H(x)) = i\frac{d}{d\omega}\left(\mathcal{F}(e^{-x}H(x))\right) = i\frac{d}{d\omega}\left(\frac{1}{1+i\omega}\right)$$
$$= \frac{-i^2}{(1+i\omega)^2} = \frac{1}{(1+i\omega)^2}.$$

With $s = i\omega$, we see that $s^2 + 3s + 2 = (s+1)(s+2)$, so

$$Y(\omega) = \frac{1}{(1+i\omega)^3(i\omega+2)} = \frac{1}{(1+i\omega)^3} - \frac{1}{(1+i\omega)^2} + \frac{1}{1+i\omega} - \frac{1}{2+i\omega}.$$

Note that

$$\frac{1}{(1+i\omega)^3} = \frac{i}{2}\frac{d}{d\omega}\left(\frac{1}{(1+i\omega)^2}\right) = \mathcal{F}\left(\frac{1}{2}t^2 e^{-x} H(x)\right)$$

since $\mathcal{F}(te^{-x}H(x)) = (1+i\omega)^{-2}$ be a previous argument. From a table, we also find that

$$\mathcal{F}(e^{-x}H(x)) = \frac{1}{1+i\omega} \quad \text{and} \quad \mathcal{F}(e^{-2x}H(x)) = \frac{1}{2+i\omega},$$

so by uniqueness,

$$y(x) = \frac{1}{2}x^2 e^{-x}H(x) - xe^{-x}H(x) + e^{-x}H(x) - e^{-2x}H(x) = \left(\frac{1}{2}(x^2 - 2x + 2)e^{-x} - e^{-2x}\right) H(x).$$

This function (and its derivatives) is absolutely integrable.

Answer: $y(x) = \frac{1}{2}(x^2 - 2x + 2)e^{-x} - e^{-2x}$ if $x \geq 0$ and $y(x) = 0$ if $x < 0$.

6. Let $f(t) = e^{-t^2}$, $t \in \mathbf{R}$, and $h(t) = \dfrac{1}{a^2 + t^2}$, $t \in \mathbf{R}$ and $a > 0$. Then

$$g_a(t) = (f * h)(t), \quad t \in \mathbf{R},$$

is the convolution of f and h. Since $f \in G(\mathbf{R})$ and $h \in G(\mathbf{R})$ and at least one (actually both) is bounded, it is clear that $g_a \in G(\mathbf{R})$ is bounded and continuous and that

$$\mathcal{F}(g_a)(\omega) = (\mathcal{F}(f)\,\mathcal{F}(h))(\omega) = \sqrt{\pi}e^{-\omega^2/4} \cdot \frac{\pi}{a}e^{-a|\omega|} = \frac{\pi^{3/2}}{a}\exp\left(-\frac{\omega^2}{4} - a|\omega|\right).$$

It is clear that the right-hand side belongs to $G(\mathbf{R}) \cap L^2(\mathbf{R})$, which by the Fourier inversion formula and Plancherel's theorem implies that $g_a \in L^2(\mathbf{R})$. Why? Consider this:

- Since $\mathcal{F}(g_a) \in L^1(\mathbf{R})$ and g_a is continuous, we know from the Fourier inversion theorem that $\mathcal{F}(\mathcal{F}(g_a)) = 2\pi g_a$.

- We also know that $\mathcal{F}(\mathcal{F}(g_a)) \in L^2(\mathbf{R})$ since $\mathcal{F}(g_a) \in L^2 \cap G(\mathbf{R})$ (by Plancherel's theorem).
- Therefore, $g_a \in L^2(\mathbf{R})$.

Hence we are allowed to use Plancherel's theorem directly:

$$\begin{aligned}\int_{-\infty}^{\infty} |g_a(t)|^2\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\pi^{3/2}}{a}\exp\left(-\frac{\omega^2}{4}-a|\omega|\right)\right|^2 d\omega \\ &= \frac{\pi^2}{2a^2}\int_{-\infty}^{\infty}\exp\left(-\frac{\omega^2}{2}-2a|\omega|\right)d\omega \le \frac{\pi^2}{2a^2}\int_{-\infty}^{\infty}\exp\left(-2a|\omega|\right)d\omega \\ &= \frac{\pi^2}{a^2}\int_0^{\infty}\exp\left(-2a\omega\right)d\omega = \frac{\pi^2}{a^2}\left[\frac{e^{-2a\omega}}{-2a}\right]_0^{\infty} = \frac{\pi^2}{2a^3}.\end{aligned}$$

Since $g_a(t) \ge 0$, this is precisely the identity we wished to prove.

Answer: $G_a(\omega) = \dfrac{\pi^{3/2}}{a}\exp\left(-\dfrac{\omega^2}{4}-a|\omega|\right)$; see above.

7. Note first that f is continuous, but not differentiable at $x = k\pi$. However, we find that $D^{\pm}f(2k\pi) = \pm 1$ and $D^{\pm}f((2k+1)\pi) = \mp 1$. Hence $f \in E'$. This is important since $y'' = 4y + f$, so we can only guarantee that $y'' \in E'$ and that y'' is continuous. This also implies that $y \in C^2$ (but we can not assume that $y \in C^3$). However, this is sufficient for the following to hold:

   - $y \in C^2$ implies that the Fourier series of both y and y' converges to $y(x)$ and $y'(x)$ respectively (Dirichlet's theorem). So, let $y(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$.
   - y being 2π-periodical and $y' \in E$ means we can form the termwise derivative of y (with equality due to the first point):

   $$y'(x) = \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}.$$

   - Similarly, $y'' \in E$ so

   $$y''(x) \sim \sum_{k=-\infty}^{\infty} -k^2 c_k e^{ikx}.$$

   - We have equality in the previous point due to the fact that $y'' \in E'$ (Dirichlet's theorem again).

   Therefore, we can write

   $$y'' - 4y = f(x) \quad\Leftrightarrow\quad \sum_{k=-\infty}^{\infty}(-k^2-4)c_k e^{ikx} = f(x).$$

   What about f? Well, $f \in E'$ is continuous so we can write $f(x)$ as its Fourier series. Let b_k be the Fourier coefficients of f:

   $$b_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}$$

and for $k \neq 0$,

$$b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| e^{-ikx} \, dx = \cdots = \frac{(-1)^k - 1}{\pi k^2}.$$

Hence

$$f(x) = \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^k - 1}{\pi k^2} e^{ikx}, \quad x \in \mathbf{R}.$$

For y to be a solution to the differential equation, we must therefore (by uniqueness) have:

$$\sum_{k=-\infty}^{\infty} (-k^2 - 4) c_k e^{ikx} = \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^k - 1}{\pi k^2} e^{ikx} \quad \Leftrightarrow \quad \begin{cases} -4c_0 = \dfrac{\pi}{2} \\ \text{and} \\ -(k^2 + 4) c_k = \dfrac{(-1)^k - 1}{\pi k^2}. \end{cases}$$

Thus $c_0 = -\frac{\pi}{8}$ and $c_k = \frac{1 - (-1)^k}{\pi k^2 (4 + k^2)}$ for $k \neq 0$. Our solution is therefore given by

$$\begin{aligned} y(x) &= -\frac{\pi}{8} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1 - (-1)^k}{\pi k^2 (4 + k^2)} e^{ikx} = -\frac{\pi}{8} + \sum_{k=1}^{\infty} \frac{1 - (-1)^k}{\pi k^2 (4 + k^2)} \left(e^{ikx} + e^{-ikx} \right) \\ &= -\frac{\pi}{8} + \sum_{k=1}^{\infty} \frac{1 - (-1)^k}{\pi k^2 (4 + k^2)} 2 \cos kx = -\frac{\pi}{8} + \sum_{k=0}^{\infty} \frac{4 \cos((2k+1)x)}{\pi (2k+1)^2 (4 + (2k+1)^2)}. \end{aligned}$$

Could you verify directly that this is the solution? Are you allowed to differentiate termwise?

Answer: $y(x) = -\frac{\pi}{8} + \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2 (4 + (2k+1)^2)}$.