Transform theory 2021-08-19 – Solutions

1. (a) No. The function is not continuous (at $\omega = 0$), so it cannot be the Fourier transform of an integrable function.
   (b) Yes. The function belongs to E, so it has a Fourier series. This series is convergent since $f \in E'$ (f has one-sided derivatives at every point). Note that the question isn't if the Fourier series converges *to* f, but only if the series is convergent.
   (c) Yes. The Laplace transform is defined by an integral, so it doesn't matter what happens at individual points.
   (d) Yes. For instance $f(x) = 1/\sqrt{|x|}$, $0 < x < 1$, and $f(x) = 0$, $x \geq 1$.
   (e) Yes. For instance $f(x) = 1/(1+x)$.

   Answer: No, Yes, Yes, Yes, Yes.

2. Clearly $f \in E$. Since $|x|$ is even, we find that

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos kx \, dx = \frac{2}{\pi}\int_0^{\pi} x \cos kx \, dx \\ &= \frac{2}{\pi}\left(\left[\frac{x \sin kx}{k}\right]_0^{\pi} - \frac{1}{k}\int_0^{\pi} \sin kx \, dx\right) = \frac{2}{\pi}\left(0 - \left[-\frac{\cos kx}{k^2}\right]_0^{\pi}\right) \\ &= \frac{2((-1)^k - 1)}{\pi k^2} \end{aligned}$$

and

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \, dx = \pi.$$

Hence

$$f(x) \sim \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k - 1}{k^2} \cos kx = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2}.$$

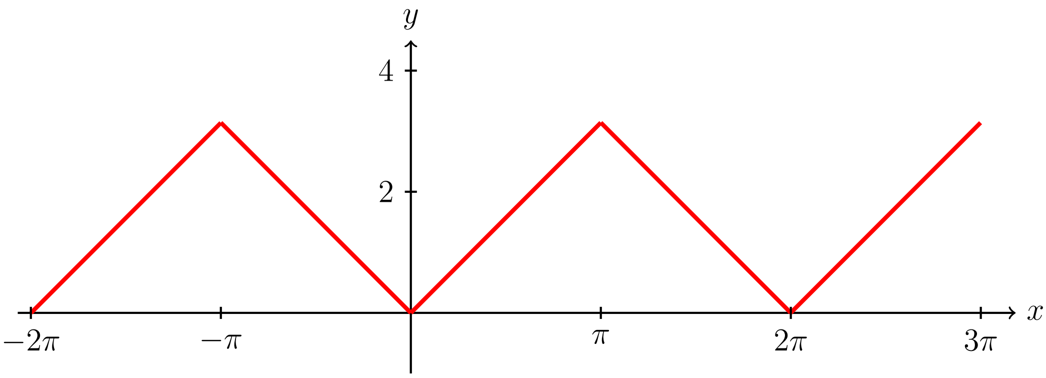
Furthermore, the function is differentiable for $x \neq n\pi$, $n \in \mathbf{Z}$ and the function has right- and lefthand derivatives at $x = n\pi$. Hence – by Dirichlet's theorem – the Fourier series is convergent and converges to $f(x)$ (due to continuity) for every $x \in \mathbf{R}$.

Since $f' \in E$ (the derivative is piecewise constant), $f(-\pi) = f(\pi)$ and f is continuous, we know that the Fourier series converges uniformly (by theorem; see Lecture 4). We can also see this by observing that

$$\left|\frac{1}{(2m+1)^2} \cos(2m+1)x\right| \leq \frac{1}{(2m+1)^2} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{1}{(2m+1)^2} < \infty,$$

so by Weierstrass' M-test we obtain uniform convergence for the Fourier series.

With this information, we can draw the graph of the Fourier series (which turns out to be identical to the graph of the function).



Answer: $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\displaystyle\sum_{m=0}^{\infty} \dfrac{\cos(2m+1)x}{(2m+1)^2}$; convergence is uniform. Graph above.

3. We're looking for exponentially bounded solutions, so we assume that $|y(t)| \leq Ke^{at}$ for some constants $a, C > 0$. The integral is the unilateral convolution of y with the function $u \mapsto e^{2u}$. Taking the Laplace transform, we obtain that

$$sY(s) - y(0) + \frac{Y(s)}{s-2} = 0 \quad \Leftrightarrow \quad Y(s)\left(s + \frac{1}{s-2}\right) = y(0), \quad \operatorname{Re} s > 2.$$

We let $y(0) = C$ (an unknown constant). Then

$$Y(s) = C\frac{s-2}{s^2 - 2s + 1} = C\frac{s-2}{(s-1)^2} = C\left(\frac{1}{s-1} - \frac{1}{(s-1)^2}\right)$$

where $\operatorname{Re} s > 1$. From a table, we now find that

$$y(t) = C(1-t)e^t, \quad t \geq 0.$$

Answer: $y(t) = C(1-t)e^t, \quad t \geq 0.$

4. We're looking for a solution to $y''(x) = 4y(x + \pi/2)$, so obviously y must be (at least) differentiable. Hence y is continuous. This means that y'' must be continuous (since y solves the equation). Hence $y \in C^2$. Which means that $y'' \in C^2$, so $y \in C^4$ and so on. In other words, the solution must be very smooth.

- $y \in C^3$ implies that the Fourier series of y, y' and y'' converges to $y(x)$, $y'(x)$ and $y''(x)$, respectively (Dirichlet's theorem). So, let $y(x) = \displaystyle\sum_{k=-\infty}^{\infty} c_k e^{ikx}$.
- y being 2π-periodical and $y' \in E$ means we can form the termwise derivative of y (with equality due to the first point):

$$y'(x) = \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}.$$

- Similarly, $y'' \in E$, so (with equality since $y \in C^3$)

$$y''(x) = \sum_{k=-\infty}^{\infty} -k^2 c_k e^{ikx}.$$

Therefore, we can write

$$y''(x) - 4y(x + \pi/2) = 0 \quad \Leftrightarrow \quad \sum_{k=-\infty}^{\infty} (-k^2 - 4e^{ik\pi/2})c_k e^{ikx} = 0.$$

For y to be a solution to the differential equation, we must therefore (by uniqueness) have:

$$k^2 = -4e^{ik\pi/2} \quad \text{or} \quad c_k = 0.$$

We see that when $k = 0, \pm 1$, we must have $c_k = 0$. For $k = \pm 2$ however, we have $k^2 = 4 = -4e^{\pm i\pi} = 4$, so $c_{\pm 2}$ are arbitrary. För $|k| \geq 3$, we find that $k^2 \geq 9$ but $|-4e^{ik\pi/2}| = 4$ so $c_k = 0$. Hence our solutions must have the form

$$y(x) = c_{-2}e^{-i2x} + c_2 e^{i2x} = A\cos 2x + B\sin 2x.$$

Answer: $y(x) = A\cos 2x + B\sin 2x$.

5. Taking the Z transform, we find that

$$zY(z) - zy[0] - Y(z) = \frac{z^4 + 4z^3 + z^2}{(z-1)^4}, \quad |z| > 1,$$

since $\mathcal{Z}(k^3) = (z^3 + 4z^2 + z)/(z-1)^4$ (from the table). Thus

$$Y(z) = \frac{z^4 + 4z^3 + z^2}{(z-1)^5} = \frac{1}{z}\frac{z^5}{(z-1)^5} + \frac{4}{z^2}\frac{z^5}{(z-1)^5} + \frac{1}{z^3}\frac{z^5}{(z-1)^5}, \quad |z| > 1.$$

From a table, we find that $\mathcal{Z}\left(\binom{k+m}{m}\right) = \dfrac{z^{m+1}}{(z-1)^{m+1}}$ for $|z| > 1$. By uniqueness, we therefore have

$$y[k] = H[k-1]\binom{k+3}{4} + 4H[k-2]\binom{k+2}{4} + H[k-3]\binom{k+1}{4}.$$

So thus we obtain that $y[0] = 0$ (as expected) and $y[1] = 1$, $y[2] = 9$. For $k \geq 3$,

$$\begin{aligned}
y[k] &= \binom{k+3}{4} + 4\binom{k+2}{4} + \binom{k+1}{4} \\
&= \frac{(k+3)!}{4!\,(k-1)!} + 4\frac{(k+2)!}{4!\,(k-2)!} + \frac{(k+1)!}{4!\,(k-3)!} \\
&= \frac{(k+3)(k+2)(k+1)k}{4!} + 4\frac{(k+2)(k+1)k(k-1)}{4!} + \frac{(k+1)k(k-1)(k-2)}{4!} \\
&= \frac{k(k+1)}{4!}\left((k+3)(k+2) + 4(k+2)(k-1) + (k-1)(k-2)\right) \\
&= \frac{k(k+1)}{4!}\left(6k^2 + 6k\right) = \frac{k^2(k+1)^2}{4}.
\end{aligned}$$

Note that this formula also holds for $k = 0$, $k = 1$ and $k = 2$, so the answer is

$$y[k] = \frac{k^2(k+1)^2}{4}, \quad k = 0, 1, 2, \ldots$$

To find the sum, consider $S_n = \sum_{m=0}^{n} m^3$ for $n = 0, 1, 2, \ldots$. Then $s_{n+1} - s_n = (n+1)^3$. By uniqueness, it follows that $s_n = y[n]$. Alternatively, we can consider that the sum is a convolution

$$u[k] = \sum_{m=0}^{k} m^3 = (1 * m^3)[k]$$

and take the Z transform: $U(z) = \mathcal{Z}(1)\,\mathcal{Z}(m^3) = \dfrac{z^4 + 4z^3 + z^2}{(z-1)^5}$. This results in the same expression as above, so $u[k] = y[k]$.

Answer: $y[k] = \displaystyle\sum_{m=0}^{k} m^3 = \frac{k^2(k+1)^2}{4}$, $k = 0, 1, 2, \ldots$

6. Since $|\cos nx| \leq 1$ for $x \in \mathbf{R}$, it is clear that

$$|f(x)| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

so the function $f(x)$ is defined for every $x \in \mathbf{R}$. Furthermore, since $\left|\dfrac{\cos nx}{n^2}\right| \leq \dfrac{1}{n^2}$ it also follows from Weierstrass M-test that the convergence is uniform. Since the functions we are summing are continuous and the convergence is uniform, it therefore follows that $f(x)$ is continuous. Hence the Riemann integral of $f(x)$ on $[0, \pi]$ exists. Moreover, again due to the uniform convergence, we can integrate the series termwise:

$$\int_0^{\pi} f(x)\,dx = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{\pi} \cos nx\,dx.$$

Note now that

$$\int_0^{\pi} \cos nx\,dx = 0, \quad n = 1, 2, 3, \ldots$$

so therefore we conclude that $\displaystyle\int_0^{\pi} f(x)\,dx = 0$.

Answer: see above; 0.

7. Since f is absolutely integrable and $|\sin \omega x| \leq 1$, we can consider the principal value. Therefore it is rather immediate that an even function will cause the integral to be zero. However, this isn't sufficient to prove the "only if" part. By Euler's formulas, we find that

$$\int_{-R}^{R} f(x) \sin \omega x\,dx = \frac{1}{2i} \int_{-R}^{R} f(x)\left(e^{i\omega x} - e^{-i\omega x}\right)\,dx.$$

Note that

$$\int_{-R}^{R} f(x) e^{i\omega x}\,dx = \Big/ t = -x \Big/ = -\int_{R}^{-R} f(-t) e^{-i\omega t}\,dt = \int_{-R}^{R} f(-t) e^{-i\omega t}\,dt.$$

Hence

$$\frac{1}{2i} \int_{-R}^{R} f(x)\left(e^{i\omega x} - e^{-i\omega x}\right)\,dx = \frac{1}{2i} \int_{-R}^{R} \big(f(-x) - f(x)\big) e^{-i\omega x}\,dx.$$

Let $g(x) = f(-x) - f(x)$. Then g is absolutely integrable and differentiable and

$$\lim_{R\to\infty} \frac{1}{2i} \int_{-R}^{R} \bigl(f(-t) - f(t)\bigr) e^{-i\omega x}\, dx = \frac{G(\omega)}{2i},$$

where G is the Fourier transform of g. By Fourier inversion (which is OK since g is differentiable), we see that

$$g(x) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} G(\omega) e^{i\omega t}\, d\omega,$$

so if $G(\omega) = 0$ for every ω, then $g(x) = 0$ is also necessary. Hence $f(-x) = f(x)$, which means that f is an even function.

Answer: see above.