Transform theory 2021-10-22 – Solutions

1. (a) Yes, $\|\mathcal{F}u\|_\infty \le \|u\|_1 < \infty$.
   (b) No, since $\lim_{|z|\to\infty}\cos(|z|)$ doesn't exist (which contradicts that the limit should be $u[0]$).
   (c) No, this is impossible. The exponent t^2 grows too fast (can't be bounded by kt for any $k > 0$).
   (d) Yes. The function belongs to $L^1(-\pi,\pi)$ (since it is continuous on $[-\pi,\pi]$), so it has a Fourier series. Moreover, the function has right- and lefthand derivatives at every point, so the Fourier series is convergent for every $x \in \mathbf{R}$.
   (e) Since y is continuous and $y''(x) = -ay(x+v)$, then y'' is also continuous. Therefore it must be true that $y \in C^2$. Hence y'' belongs to C^2, so $y \in C^4$. This does not hold for $y(x) = |x|^3$ (the third derivative contains a jump).

   Answer: Yes, No, No, Yes, No.

2. Let $v[k] = (k^2+3k)2^k$, $k = 0,1,2,\ldots$. Then

$$f[n] = \sum_{k=0}^{n} v[k] = (1 * v)[n]$$

is the convolution of v with 1. Hence

$$F(z) = \mathcal{Z}(1)\,\mathcal{Z}(v) = \frac{z}{z-1}\,\mathcal{Z}(v),$$

an identity you also can find in the collection of formulas. To find the Z-transform of v, we then use the linearity and a table:

$$\mathcal{Z}(v) = \mathcal{Z}(k^2 2^k) + 3\,\mathcal{Z}(k2^k) = \frac{2z^2+4z}{(z-2)^3} + \frac{6z}{(z-2)^2} = \frac{2z(z+2+3(z-2))}{(z-2)^3}$$
$$= \frac{8z(z-1)}{(z-2)^3}$$

for $|z| > 2$. Therefore,

$$F(z) = \frac{z}{z-1}\,\frac{8z(z-1)}{(z-2)^3} = \frac{8z^2}{(z-2)^3} = \frac{8}{z}\,\frac{z^3}{(z-2)^3}, \quad |z| > 2.$$

We find from a table that $\mathcal{Z}\left(\binom{k+m}{m} a^k\right) = \dfrac{z^{m+1}}{(z-a)^{m+1}}$ for $|z| > a$, so by uniqueness,

$$f[n] = 8\binom{(n+2)-1}{2} 2^{n-1} H[n-1] = \binom{n+1}{2} 2^{n+2} = (n^2+n)2^{n+1}, \quad n = 1,2,\ldots,$$

and $f[0] = 0$.

Answer: $(n^2+n)2^{n+1}$, $n = 0,1,2,3,\ldots$.

3. Clearly $u \in E$. We find that, for $k \neq 0$,

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-ikx}\,dx = \frac{1}{2\pi}\left[\frac{e^{-ikx}}{-ik}\right]_{-\pi/2}^{\pi/2} \\ &= \frac{1}{k\pi}\,\frac{e^{ik\pi/2} - e^{-ik\pi/2}}{2i} = \frac{\sin(k\pi/2)}{k\pi} \end{aligned}$$

and

$$c_0 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} dx = \frac{1}{2}.$$

Hence

$$\begin{aligned} u(x) &\sim \frac{1}{2} + \sum_{k\neq 0} \frac{\sin(k\pi/2)}{k\pi}\, e^{ikx} = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\sin(k\pi/2)}{k}\cos kx \\ &= \frac{1}{2} + \frac{2}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m \cos(2m+1)x}{2m+1} \end{aligned}$$

Furthermore, the function is differentiable for $x \neq (m+1/2)\pi$, $m \in \mathbf{Z}$ and the function has right- and lefthand derivatives at $x = (m+1/2)\pi$. Hence – by Dirichlet's theorem – the Fourier series is convergent and converges to $u(x)$ for every $x \neq (m+1/2)\pi$ and to $1/2$ for $x = (m+1/2)\pi$.

Using Parseval's identity, that is,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|^2\,dx = \sum_{k=-\infty}^{\infty} |c_k|^2,$$

we find that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|^2\,dx = \frac{1}{2\pi}\cdot\pi = \frac{1}{2}$$

so

$$\begin{aligned} \frac{1}{2} &= \sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{1}{4} + \sum_{k\neq 0} \frac{|\sin(k\pi/2)|^2}{\pi^2 k^2} \\ &= \frac{1}{4} + 2\sum_{k=1}^{\infty} \frac{|\sin(k\pi/2)|^2}{\pi^2 k^2} = \frac{1}{4} + \frac{2}{\pi^2}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}. \end{aligned}$$

Rearranging this yields

$$\frac{1}{2} - \frac{1}{4} = \frac{2}{\pi^2}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \quad \Leftrightarrow \quad \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$

Answer: $u(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m \cos(2m+1)x}{2m+1}$; $\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}$.

4. We're looking for exponentially bounded solutions, so we assume that $|y(t)| \leq Ke^{at}$ for some constants $a, C > 0$. Taking the Laplace transform, we obtain that

$$sY(s) - y(0) - 2Y(s) = G(s) \quad \Leftrightarrow \quad Y(s)\,(s-2) = G(s),$$

where $G(s)$ is the Laplace transform of the right-hand side. For this to make sense, we assume that y' is piecewise continuous (so we're not assuming that $y \in C^1$). We then calculate $G(s)$ directly by using the definition:

$$
\begin{aligned}
G(s) &= \int_0^1 4te^{-st}\,dt + \int_1^\infty 4(3-t)e^{-st}\,dt \\
&= \left[\frac{4te^{-st}}{-s}\right]_0^1 + \frac{4}{s}\int_0^1 e^{-st}\,dt + \left[\frac{4(3-t)e^{-st}}{-s}\right]_1^\infty - \frac{4}{s}\int_1^\infty e^{-st}\,dt \\
&= -\frac{4e^{-s}}{s} + \left[\frac{4e^{-st}}{-s^2}\right]_0^1 + \frac{8e^{-s}}{s} - \left[\frac{4e^{-st}}{-s^2}\right]_1^\infty \\
&= \frac{4e^{-s}}{s} - \frac{4e^{-s}}{s^2} + \frac{4}{s^2} - \frac{4e^{-s}}{s^2} = \frac{4e^{-s}}{s} - \frac{8e^{-s}}{s^2} + \frac{4}{s^2},
\end{aligned}
$$

where we assume that $\operatorname{Re} s > 0$.

Then

$$
Y(s) = \frac{1}{s-2}\left(\frac{4}{s} - \frac{8}{s^2}\right)e^{-s} + \frac{4}{s^2(s-2)} = \frac{4}{s^2}\,e^{-s} + \frac{1}{s-2} - \frac{1}{s} - \frac{2}{s^2},
$$

where $\operatorname{Re} s > 2$. From a table, we now find that

$$
\mathcal{L}\,(4t) = \frac{4}{s^2}
$$

and

$$
\mathcal{L}\left(e^{2t} - 1 - 2t\right) = \frac{1}{s-2} - \frac{1}{s} - \frac{2}{s^2}.
$$

The factor e^{-s} causes a time-shift, so we obtain that

$$
y(t) = 4(t-1)H(t-1) + e^{2t} - 1 - 2t, \quad t \geq 0.
$$

We note that $y(1^-) = y(1^+)$, so y is at least continuous. However,

$$
D^- y(1) = 2e^{2t} - 2\big|_{t=1} = 2e^2 - 2
$$

and

$$
D^+ y(1) = 4 + 2e^{2t} - 2\big|_{t=1} = 2e^2 + 2,
$$

so y' is not continuous (in fact $y'(1)$ can be seen to not exist by considering the difference quotient directly). Clearly though, $D^\pm y(1) - 2y(1^\pm) = \mathrm{RHS}(1^\pm)$, so in that sense we have a solution also in $t = 1$ (but not in the classical sense).

Answer: $y(t) = 4(t-1)H(t-1) + e^{2t} - 1 - 2t, \quad t \geq 0$. Not in C^1. See above.

5. Using Euler's identity and changing the integration variable (let $\xi = -x$) in one of the resulting integrals shows that

$$
\begin{aligned}
\int_0^\infty \frac{\cos tx}{4+x^2}\,dx &= \frac{1}{2}\int_0^\infty \frac{1}{4+x^2}\,e^{ixt}dx + \frac{1}{2}\int_0^\infty \frac{1}{4+x^2}\,e^{-ixt}dx \\
&= -\frac{1}{2}\int_0^{-\infty} \frac{1}{4+(-\xi)^2}\,e^{i(-\xi)t}d\xi + \frac{1}{2}\int_0^\infty \frac{1}{4+x^2}\,e^{-ixt}dx \\
&= \frac{1}{2}\int_{-\infty}^0 \frac{1}{4+\xi^2}\,e^{-i\xi t}d\xi + \frac{1}{2}\int_0^\infty \frac{1}{4+x^2}\,e^{-ixt}dx \\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{1}{4+x^2}\,e^{-ixt}dx,
\end{aligned}
$$

which we recognize as the Fourier transform

$$\frac{1}{2}\,\mathcal{F}\left(\frac{1}{4+x^2}\right)(t), \quad t \in \mathbf{R}.$$

From a table, it therefore follows that

$$\int_0^\infty \frac{\cos tx}{4+x^2}\,dx = \frac{\pi}{4}\,e^{-2|t|}.$$

Answer: $\frac{\pi}{4}\,e^{-2|t|}$.

6. With these Fourier coefficients, we find that

$$g(x) = \sum_{n=0}^{\infty} e^{-n}\,e^{inx} = \sum_{n=0}^{\infty}\left(e^{-1+ix}\right)^n = \frac{1}{1-e^{-1+ix}} = \frac{e}{e-e^{ix}}$$

since the series is geometric. We would expect that the function in the right-hand side has the desired properties and want to conclude that $f(x) = g(x)$. Since $|e^{inx}| \le 1$ for $x \in \mathbf{R}$, it is clear that

$$|g(x)| \le \sum_{n=0}^{\infty} e^{-n}\left|e^{inx}\right| = \sum_{n=0}^{\infty} e^{-n} = \frac{1}{1-e^{-1}} < \infty,$$

so the function $g(x)$ is defined for every $x \in \mathbf{R}$. Furthermore, it also follows from Weierstrass M-test that the convergence is uniform. Since the functions we are summing are continuous and the convergence is uniform, it therefore follows that $g(x)$ is continuous. Hence the Riemann integral of $f(x)$ on $[-\pi, \pi]$ exists. Moreover, again due to the uniform convergence, we can integrate the series termwise to obtain the Fourier coefficients of g:

$$d_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,e^{-ikx}\,dx = \sum_{n=0}^{\infty}\frac{e^{-n}}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,e^{-ikx}\,dx.$$

Note now that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-k)x}\,dx = \begin{cases} 0, & n \ne k, \\ 1, & n = k \end{cases},$$

so therefore we conclude that $d_k = e^{-k}$ for $k = 0, 1, 2, \ldots$ and $d_k = 0$ for $k < 0$. Hence g is continuous and has the same Fourier coefficients as f, so by uniqueness it follows that $f(x) = g(x)$ for all $x \in \mathbf{R}$.

Answer: see above; $f(x) = \dfrac{e}{e-e^{ix}}$.

7. We're looking for a continuous function $u \in L^1(\mathbf{R})$ where $\mathcal{F}u \notin L^1(\mathbf{R})$. First, observe that if $\mathcal{F}u \in L^1(\mathbf{R})$, then $\mathcal{F}(\mathcal{F}(u))(x) = 2\pi u(-x)$. By the Riemann-Lebesgue lemma, this implies that $u(x) \to 0$ as $x \to \pm\infty$. So it is necessary that u tends to zero for $\mathcal{F}u$ to belong to $L^1(\mathbf{R})$ (note however that it is **not** sufficient). Hence if we can construct a continuous function $u \in L^1(\mathbf{R})$ such that $\lim_{x\to\infty} u(x) \ne 0$, then we're done.

Now let's consider a special function. Define

$$v(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ 2-2x, & \frac{1}{2} \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Then

$$\int_{-\infty}^{\infty} v(x)\,dx = \frac{1}{2}.$$

For $k = 0, 1, 2, \ldots$, let

$$u_k(x) = v\left(2^k(x-k)\right), \quad x \in \mathbf{R}.$$

The sequence u_k consists of shifted and scaled triangles with area

$$\int_{-\infty}^{\infty} u_k(x)\,dx = \Big/ \ t = 2^k(x-k) \ \Big/ = 2^{-k}\int_{-\infty}^{\infty} u(x)\,dx = 2^{-k-1}.$$

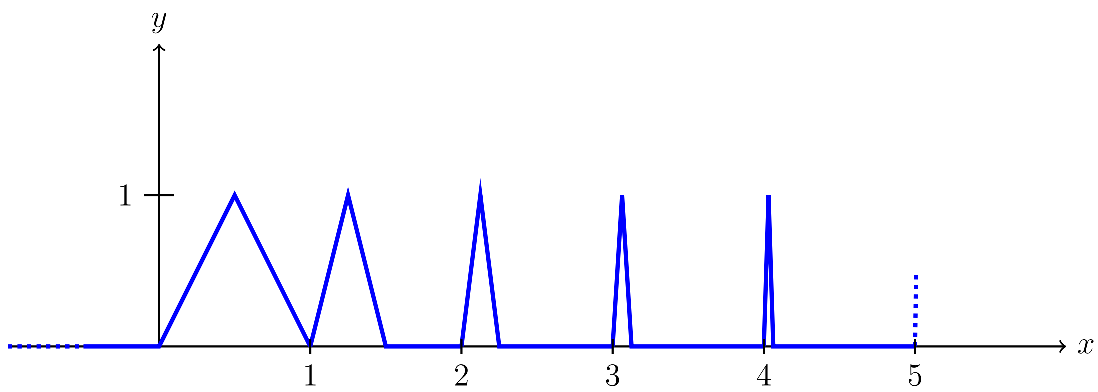
Now, let us define

$$u(x) = \sum_{k=0}^{\infty} u_k(x), \quad x \in \mathbf{R}.$$

Then $u(x)$ exists as a nonnegative number for every x (the partial sums are increasing and bounded) and

$$\int_{-\infty}^{\infty} u(x)\,dx = \sum_{k=0}^{\infty} 2^{-k-1} = 2^{-1}\frac{1}{1-2^{-1}} = 1,$$

so $u \in G(\mathbf{R})$. It is also clear that u is continuous (you can prove this directly but it is clear from the construction).



If we choose the sequence $x_k = k$, then $u(x_k) = 1$ so it is impossible that $u(x) \to 0$ as $x \to \infty$. Therefore it follows by the argument above that $\mathcal{F}\,u \notin L^1(\mathbf{R})$.

Answer: see above.