LINKÖPINGS UNIVERSITET
Matematiska institutionen
Transform theory

Utbildningskod: TATA57
Modul: TEN1
Datum: 2022-06-03
Institution: MAI

Exam in Transform Theory

TATA57/TEN1 2022-06-03

You are permitted to use:

- Transformteori - Sammanfattning, Formler & Lexikon (from MAI);
- Table of Formulæ (by Johan Thim). This one is included at the end of the exam.
- A calculator (with cleared memories).

Swedish version of the exam comes after the English version. You may answer in Swedish or English.

Each problem is worth 5 points (for a total of 35 points). No half-points will be awarded in the grading. A solution is deemed good if the question is awarded at least 3 points out of 5. For grade n, you need at least n good answers.

ERASMUS students will have their grades marked according to the scale: A = grade 5, B = grade 4, C = grade 3.

Grading (sufficient limit): 15 points for grade 3. Your solutions need to be complete, well motivated, carefully written and concluded with a clear answer. Be very careful with motivations since these are a huge part of your solutions. Make sure to point out that conditions in theorems you are using hold. Assumptions you make need to be explicit. The exercises are not necessarily in order of difficulty.

Solutions can be found on the homepage a couple of hours after the finished exam.

1. For each of these statements, prove or disprove the claims. For each correct answer, you get 1 p and for each incorrect answer, you get 0 p.

   (a) The function $F(\omega) = \dfrac{1}{\omega^2 - 1}$ is the Fourier transform of a function $f \in G(\mathbf{R})$.

   (b) The function $F(\omega) = e^{-|\omega|}$ is the Fourier transform of a function $f \in G(\mathbf{R})$.

   (c) A function $u \in L^1(\mathbf{R})$ also belongs to $L^2(\mathbf{R})$.

   (d) The function $F(s) = \dfrac{s}{s-1}$, $\operatorname{Re} s > 1$, is the unilateral Laplace transform of an exponentially bounded function $u : [0, \infty[\to \mathbf{C}$.

   (e) Let $u_n : [0,1] \to \mathbf{R}$ be a sequence of continuous functions such that $\lim_{n\to\infty} u_n(x) = u(x)$ exists for $0 \le x \le 1$. If $\lim_{n\to\infty} \int_0^1 u_n(x)\,dx = \int_0^1 u(x)\,dx$, then u_n converges uniformly to u on $[0,1]$.

2. (a) Find the Laplace transform of $u(t) = (3-t)H(3-t) = \begin{cases} 3-t, & 0 \le t \le 3, \\ 0, & t > 3. \end{cases}$ (2p)

   (b) Solve the equation $y''(t) + 4y(t) = 4(3-t)H(3-t)$, $y(0) = 0$, $y'(0) = 1$, $t \ge 0$. (4p)

3. Let

$$u(x) = \begin{cases} 0, & -\pi \le x < 0, \\ x, & 0 \le x < \pi, \end{cases}$$

   and extend u periodically. Find the Fourier series for $u(x)$ and show what the Fourier series converges to for $x \in \mathbf{R}$. Draw the graph of the Fourier series. Is the convergence uniform on $[-\pi, \pi]$?

4. Solve the equation

$$u[n+1] - 2u[n] + 4\sum_{k=0}^{n} u[n-k]2^k = 0, \quad u[0] = 1, \quad n = 0, 1, 2, \ldots$$

5. Calculate

$$\int_{-\infty}^{\infty} \frac{(1-\cos 4t)^2}{t^4}\,dt.$$

   Motivate your assumptions.

6. (a) Prove that for $u \in G(\mathbf{R})$, the Fourier transform $U(\omega)$ of u is bounded on $\mathbf{R}$. (2p)

   (b) Suppose that $u' \in G(\mathbf{R})$ and that u is continuous. Prove that $\mathcal{F}(u') = i\omega\,\mathcal{F}(u)$. You may use the fact that the assumptions imply that $u(x) \to 0$ as $|x| \to \infty$ without proof. (3p)

7. Let $u(x) = \displaystyle\sum_{k=1}^{\infty} \frac{\arctan k}{k^2(k+1)} \cos\frac{kx}{2}$, $-\pi \le x \le \pi$. Show that $u(x)$ is a continuous function on $[-\pi, \pi]$ and find the Fourier coefficients for u. Prove that the Fourier series converges to $u(x)$ for all $x \in [-\pi, \pi]$. Your Fourier coefficients may contain infinite series but no unevaluated integrals. Motivate carefully!

Svensk översättning av uppgifterna

1. För varje påstående, bevisa eller motbevisa påståendet. Du får 1 p för varje rätt och 0 p för varje fel.

   (a) Funktionen $F(\omega) = \dfrac{1}{\omega^2 - 1}$ är Fouriertransformen av en funktion $f \in G(\mathbf{R})$.

   (b) Funktionen $F(\omega) = e^{-|\omega|}$ är Fouriertransformen av en funktion $f \in G(\mathbf{R})$.

   (c) En funktion $u \in L^1(\mathbf{R})$ tillhör också $L^2(\mathbf{R})$.

   (d) Funktionen $F(s) = \dfrac{s}{s-1}$, $\operatorname{Re} s > 1$, är den enkelsidiga Laplacetransformen av en exponentiellt begränsad funktion $u : [0, \infty[\to \mathbf{C}$.

   (e) Låt $u_n : [0,1] \to \mathbf{R}$ vara en följd kontinuerliga funktioner sådana att $\lim_{n\to\infty} u_n(x) = u(x)$ existerar för $0 \le x \le 1$. Om $\lim_{n\to\infty} \int_0^1 u_n(x)\,dx = \int_0^1 u(x)\,dx$, så konvergerar u_n likformigt till u på $[0,1]$.

2. (a) Bestäm Laplacetransformen av $u(t) = (3-t)H(3-t) = \begin{cases} 3-t, & 0 \le t \le 3, \\ 0, & t > 3. \end{cases}$. (2p)

   (b) Lös ekvationen $y''(t) + 4y(t) = 4(3-t)H(3-t)$, $y(0) = 0$, $y'(0) = 1$, $t \ge 0$. (4p)

3. Låt

$$u(x) = \begin{cases} 0, & -\pi \le x < 0, \\ x, & 0 \le x < \pi, \end{cases}$$

   och utvidga u periodiskt. Bestäm Fourierserien för $u(x)$ och visa vad Fourierserien konvergerar till för $x \in \mathbf{R}$. Skissa grafen för Fourierserien. Är konvergensen likformig på $[-\pi, \pi]$?

4. Lös ekvationen

$$u[n+1] - 2u[n] + 4\sum_{k=0}^{n} u[n-k]2^k = 0, \quad u[0] = 1, \quad n = 0, 1, 2, \ldots$$

5. Beräkna

$$\int_{-\infty}^{\infty} \frac{(1-\cos 4t)^2}{t^4}\,dt.$$

   Motivera dina antaganden.

6. (a) Visa att för $u \in G(\mathbf{R})$ så är Fouriertransformen $U(\omega)$ av u begränsad på $\mathbf{R}$. (2p)

   (b) Antag att $u' \in G(\mathbf{R})$ och att u är kontinuerlig. Bevisa att $\mathcal{F}(u') = i\omega\,\mathcal{F}(u)$. Du får använda det faktum att antagandena implicerar att $u(x) \to 0$ as $|x| \to \infty$ utan bevis. (3p)

7. Låt $u(x) = \sum_{k=1}^{\infty} \dfrac{\arctan k}{k^2(k+1)} \cos \dfrac{kx}{2}$, $-\pi \le x \le \pi$. Visa att $u(x)$ är en kontinuerlig funktion på $[-\pi, \pi]$ och hitta Fourierkoefficienterna för u. Bevisa att Fourierserien konvergerar till $u(x)$ för alla $x \in [-\pi, \pi]$. Dina Fourierkoefficienter får innehålla oändliga serier men inga oberäknade integraler. Motivera noggrant!

Table of formulæ

Johan Thim (johan.thim@liu.se)

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1 Notation and Definitions

- **R** is the set of all real numbers.
- **Q** is the set of all rational numbers.
- **C** is the set of all complex numbers.
- **Z** is the set of all integers.
- $\mathbf{N} = \{0, 1, 2, \ldots\}$ is the set of all natural numbers (including 0).

For $z = x + iy \in \mathbf{C}$, $x, y \in \mathbf{R}$,

$$\operatorname{Re} z = x, \quad \operatorname{Im} z = y, \quad |z| = \sqrt{x^2 + y^2}.$$

1.1 Continuity and Differentiability

- One-sided limits:
$$u(x^{\pm}) = \lim_{x \to x^{\pm}} u(x).$$
- One-sided derivatives:
$$D^{\pm}u(x) = \lim_{h \to 0^{\pm}} \frac{u(x+h) - u(x)}{h}.$$
- $C(I)$: The set of all continuous functions on a set I.
- $C^m(I)$: The set of all continuously differentiable (up to order m) functions on a set I.

A function $u\colon I \to \mathbf{C}$ on an interval I is called piecewise continuous if...

- I is finite and there are a finite number of points such that u is continuous everywhere on I except for at these points. Moreover, if $c \in I$ is a point where u is discontinuous, the limits
$$\lim_{I \ni x \to c^-} u(x) \quad \text{and} \quad \lim_{I \ni x \to c^+} u(x)$$
exist (only $u(c^-)$ or $u(c^+)$ if points on the boundary of I).
- I is infinite and there a finite number of exception points (as in the finite case) in each finite sub-interval of I.

1.2 Function Spaces

A normed linear space is a linear space V endowed with a norm $\|\cdot\|\colon V \to [0, \infty[$ such that

$$\text{(i) } \|u\| \geq 0 \qquad \text{(ii) } \|\alpha u\| = |\alpha|\|u\|,\ \alpha \in \mathbf{C} \qquad \text{(iii) } \|u + v\| \leq \|u\| + \|v\|.$$

An inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbf{C}$ on a vector space V satisfies

$$\text{(i) } \langle u, v \rangle = \overline{\langle v, u \rangle} \qquad \text{(ii) } \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \qquad \text{(iii) } \langle \alpha u, v \rangle = \alpha \langle u, v \rangle$$

$$\text{(iv) } \langle u, u \rangle \geq 0 \qquad \text{(v) } \langle u, u \rangle = 0 \quad \Leftrightarrow \quad u = 0.$$

In an inner product space, we use $\|u\| = \sqrt{\langle u, u \rangle}$ as the norm.

Sequence Spaces

The sequence spaces l^p, $1 \leq p \leq \infty$, consists of sequences $(x_0, x_1, x_2, x_3, \ldots)$, $x_i \in \mathbf{C}$, such that the norm

$$\|x\|_{l^p} = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

or

$$\|x\|_{l^\infty} = \sup_{k \geq 0} |x_k| < \infty, \quad p = \infty.$$

Sometimes we write $l^p(\mathbf{N})$. The spaces $l^p(\mathbf{Z})$ are defined analogously. Only l^2 is an inner product space with

$$\langle x, y \rangle = \sum_{k=0}^{\infty} x_k \overline{y_k}, \quad x, y \in l^2.$$

Lebesgue Spaces (integrable functions)

The space $L^1(a, b)$ of absolutely integrable functions $u\colon]a, b[\to \mathbf{C}$ with norm

$$\|f\|_{L^1(a,b)} = \int_a^b |f(x)| \, dx < \infty.$$

The space $L^2(a, b)$ consists of all "square integrable" functions with the norm

$$\|f\|_{L^2(a,b)} = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2} < \infty,$$

which is an inner product space with

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} \, dx.$$

The space $L^\infty(a, b)$ of bounded functions with norm

$$\|f\|_{L^\infty(a,b)} = \sup_{a \leq x \leq b} |f(x)| < \infty.$$

Note that $a = -\infty$ and/or $b = \infty$ is allowed (so we might write $L^p(\mathbf{R})$). Sometimes we write $\|f\|_p$ instead of $\|f\|_{L^p(a,b)}$.

Spaces of Piecewise Functions

- $E[a,b]$ (or E): The linear space of all piecewise continuous functions on an interval $[a,b]$.
- $E'[a,b]$ (or E'): The linear space of those $u \in E[a,b]$ such that $D^-u(x)$ exists for $a < x \le b$ and that $D^+u(x)$ exists for $a \le x < b$.
- $G(\mathbf{R})$ (or G): The linear space of all piecewise continuous functions on $\mathbf{R}$ that are absolutely integrable on $\mathbf{R}$.

1.3 Special Functions

- **Heaviside function**:

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \ge 0. \end{cases}$$

- **Signum function**:

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Discrete Functions

- **Discrete Heaviside function**:

$$H[k] = \begin{cases} 0, & k < 0, \\ 1, & k \ge 0. \end{cases}$$

- **Discrete impulse function**:

$$\delta[k] = \begin{cases} 0, & k \ne 0, \\ 1, & k = 0. \end{cases}$$

- **Binomial coefficient functions**:

$$\begin{pmatrix} n \\ k \end{pmatrix} = \begin{cases} \dfrac{n!}{(n-k)!\,k!}, & k = 0, 1, 2, \ldots, \\ 0, & k > n. \end{cases}$$

Convolutions (on R)

The convolution $u * v \colon \mathbf{R} \to \mathbf{C}$ of two functions $u \colon \mathbf{R} \to \mathbf{C}$ and $v \colon \mathbf{R} \to \mathbf{C}$ is defined by

$$(u * v)(x) = \int_{-\infty}^{\infty} u(t)v(x-t)\,dt, \quad x \in \mathbf{R},$$

whenever this integral exists. If $u, v \in L^1(\mathbf{R})$, then $u * v \in L^1(\mathbf{R})$. If one function is also bounded, then $u * v$ is continuous and bounded.
Suppose that $u, v, w \in G(\mathbf{R})$ and one function in each convolution is bounded. Then the convolution has the following properties.

- Associative: $((u * v) * w)(x) = (u * (v * w))(x)$.
- Distributive: $((u + v) * w)(x) = (u * w)(x) + (v * w)(x)$.
- Commutative: $(u * v)(x) = (v * u)(x)$.

Convolutions (on Z)

- For $u, v \colon \mathbf{Z} \to \mathbf{C}$, the **discrete convolution** $u * v$ is

$$(u * v)[n] = \sum_{k=-\infty}^{\infty} u[k]v[n-k], \quad n \in \mathbf{Z},$$

whenever this series exists.

- For $u, v \colon \mathbf{N} \to \mathbf{C}$, the **unilateral (or one-sided) discrete convolution** $u * v$ is

$$(u * v)[n] = \sum_{k=0}^{n} u[k]v[n-k], \quad n = 0, 1, 2, \ldots$$

1.4 Inequalities

- **The Cauchy-Schwarz inequality**: If $u, v \in V$ and V is an inner product space, then

$$|\langle u,\ v\rangle| \le \|u\|\|v\|.$$

- **Bessel's inequality**: Let V be an inner product space, let $v \in V$ and let $\{e_1, e_2, \ldots\}$ be an ON system in V. Then

$$\sum_{k=1}^{\infty} |\langle v,\ e_k\rangle|^2 \le \|v\|^2.$$

This implies the **Riemann-Lebesgue lemma** for inner product spaces:

$$\lim_{n\to\infty} \langle v,\ e_n\rangle = 0.$$

- **The triangle inequality**: In a normed space V,

$$|\|u\| - \|v\|| \le \|u + v\| \le \|u\| + \|v\|.$$

- **Young's inequality** ($r = p = q = 1$):

$$\|u * v\|_{L^1(\mathbf{R})} \le \|u\|_{L^1(\mathbf{R})}\|v\|_{L^1(\mathbf{R})}.$$

and

$$\|u * v\|_{l^1(\mathbf{Z})} \le \|u\|_{l^1(\mathbf{Z})}\|v\|_{l^1(\mathbf{Z})}.$$

1.5 Convergence of Sequences

Let $u_1, u_2, \ldots$ be a sequence in a normed space V. We say that $u_n \to u$ for some $u \in V$ if $\|u_n - u\| \to 0$ as $n \to \infty$. This is called **strong convergence** or **convergence in norm**.

Convergence of Functions

- **Pointwise convergence**: We say that $u_k \to u$ pointwise on I as $k \to \infty$ if
$$\lim_{k\to\infty} u_k(x) = u(x)$$
for every $x \in I$. We often refer to u as the *limiting function*.
- **Uniform convergence**: We say that $u_k \to u$ uniformly on $[a,b]$ as $k \to \infty$ if
$$\lim_{k\to\infty} \|u_k - u\|_{L^\infty(a,b)} = 0.$$

Weierstrass' M-test: If $I \subset \mathbf{R}$ and M_k, $k = 1,2,\ldots$, are constants such that $|u_k(x)| \le M_k$ for $x \in I$, then
$$\sum_{k=1}^{\infty} M_k < \infty \quad \Rightarrow \quad \sum_{k=1}^{\infty} u_k(x) \text{ converges uniformly on } I.$$

If:

- $u_0, u_1, u_2, \ldots$ are continuous functions on $[a,b]$
- and $u(x) = \sum_{k=0}^{\infty} u_k(x)$ is uniformly convergent for $x \in [a,b]$,

then

- the series u is a continuous function on $[a,b]$,
- we can exchange the order of summation and integration:
$$\int_c^d u(x)\,dx = \int_c^d \left(\sum_{k=0}^{\infty} u_k(x)\right) dx = \sum_{k=0}^{\infty} \int_c^d u_k(x)\,dx, \quad \text{for } a \le c < d \le b,$$
- and if in addition $\sum_{k=0}^{\infty} u_k'(x)$ converges uniformly on $[a,b]$, then
$$u'(x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} u_k(x)\,dx\right) = \sum_{k=0}^{\infty} \frac{d}{dx} u_k(x) = \sum_{k=0}^{\infty} u_k'(x), \quad x \in [a,b].$$

1.6 Integration Theory

The **principal value** integral is defined by
$$\text{P.V.} \int_{-\infty}^{\infty} u(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} u(x)\,dx.$$

- If $F(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ exists for every $x \in I$ and
$$\sup_{x\in I} \left| \int_{-R}^{R} f(x,y)\,dy - F(x) \right| \to 0, \text{ as } R \to \infty,$$
then we call the integral defining $F(x)$ uniformly convergent on I.

- **Dominated convergence**:
If:
  - $f\colon \mathbf{R}^2 \to \mathbf{C}$,
  - $F(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ exists for all x,
  - there is a $g \in L^1(\mathbf{R})$ such that $|f(x,y)| \le g(y)$ for all $x, y \in \mathbf{R}$,

  then $\int_{-\infty}^{\infty} f(x,y)\,dy$ converges uniformly on $\mathbf{R}$.
- **Continuity**: If $f\colon \mathbf{R}^2 \to \mathbf{C}$ is continuous on $[c,d] \times [a,R]$. Then
  - $F_R(x) = \int_a^R f(x,y)\,dy$ is continuous on $[c,d]$
  - and if in addition f is continuous on $[c,d] \times [a,\infty[$ and $F(x) = \int_a^{\infty} f(x,y)\,dy$ converges uniformly (on $[c,d]$), then F is continuous.
- **Order of integration**: If $f\colon \mathbf{R}^2 \to \mathbf{C}$ is continuous on $[c,d] \times [a,\infty[$ and $F(x)$ converges uniformly (on $[c,d]$), then
$$\int_c^d \left(\int_a^{\infty} f(x,y)\,dy\right) dx = \int_a^{\infty} \left(\int_c^d f(x,y)\,dx\right) dy.$$
- Note that we can let $a = -\infty$ in the previous theorems by exchanging $[a,R]$ by $[-R,R]$ and consider the principal values.
- **Leibniz's rule**: If
  - $f\colon \mathbf{R}^2 \to \mathbf{C}$ and $f_x'(x,y)$ exist and are continuous,
  - $F(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ is convergent for every x,
  - and $\int_{-\infty}^{\infty} f_x'(x,y)\,dy$ is uniformly convergent,

  then
$$F'(x) = \frac{d}{dx} \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\infty}^{\infty} f_x'(x,y)\,dy.$$

2 Fourier Series

For $u \in L^1(-\pi,\pi)$:
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\cos kx\,dx \quad \text{and} \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\sin kx\,dx \quad \text{or} \quad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx$$
are the Fourier coefficients (real or complex) for u. The series
$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\big(a_k \cos kx + b_k \sin kx\big) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$
is called the **Fourier series** of the function u (real or complex). We write $u(x) \sim S(x)$.
Note that:

- if u is even, then $b_k = 0$ for $k = 1, 2, 3, \ldots$;
- if u is odd, then $a_k = 0$ for $k = 1, 2, 3, \ldots$.

If u is a T-periodic function, we define $\Omega = \dfrac{2\pi}{T}$. The real Fourier series of u is then given by

$$u(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \big(a_k \cos k\Omega x + b_k \sin k\Omega x \big),$$

where

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} u(x) \cos k\Omega x \, dx \quad \text{and} \quad b_k = \frac{2}{T} \int_{-T/2}^{T/2} u(x) \sin k\Omega x \, dx.$$

The complex Fourier series is given by

$$u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ik\Omega x}, \quad \text{where } c_k = \frac{1}{T} \int_{-T/2}^{T/2} u(x) e^{-ik\Omega x} \, dx.$$

Sometimes we denote $c_k = \widehat{u}[k]$.

2.1 Parseval's identity

- **Parsevals's identity**:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(x)|^2 \, dx = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad \text{or} \quad \frac{1}{\pi} \int_{-\pi}^{\pi} |u(x)|^2 \, dx = \frac{|a_0|^2}{2} + \sum_{k=1}^{\infty} \big(|a_k|^2 + |b_k|^2 \big),$$

  where $u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ (or the real counterpart).

- **Parseval's generalized identity**:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) \overline{v(x)} \, dx = \sum_{k=-\infty}^{\infty} c_k \overline{d_k},$$

  where $u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ and $v(x) \sim \sum_{k=-\infty}^{\infty} d_k e^{ikx}$.

2.2 Convergence

Kernels

- The **Dirichlet kernel**: $D_n(x) = \sum_{k=-n}^{n} e^{ikx}$, $x \in \mathbf{R}$, $n = 1, 2, 3, \ldots$
- The **Fejér kernel**: $F_n(x) = \dfrac{1}{n+1} \sum_{l=0}^{n} \sum_{k=-l}^{l} e^{ikx} = \sum_{k=-n}^{n} \left(1 - \dfrac{|k|}{n+1} \right) e^{ikx}, \quad n = 0, 1, 2, \ldots$

2.3 Convergence Results

- If $u \in L^1(-\pi, \pi)$, then u has a Fourier series.
- Let $u \in E'$. Then

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} \to \frac{u(x^+) + u(x^-)}{2}, \quad x \in [-\pi, \pi].$$

- If $u \in E$ and $D^{\pm} u(x)$ exists, then

$$\lim_{n \to \infty} S_n(x) = \frac{u(x^+) + u(x^-)}{2}.$$

- If $\sum_{k=-\infty}^{\infty} |c_k| < \infty$, then $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ converges uniformly.
- If $u \in E$, then the Fejér means $\overline{S_n}(x) \to \dfrac{u(x^+) + u(x^-)}{2}$.
- If $u, v \in E$ and $\widehat{u}[k] = \widehat{v}[k]$, $k \in \mathbf{Z}$, then $u(x) = v(x)$ whenever u and v are continuous at x.
- If $u' \in E$, u is continuous and $u(-\pi) = u(\pi)$, then $S_n(x)$ converges uniformly to $u(x)$.
- If $u' \in E$ and u is continuous on $[a, b] \subset]-\pi, \pi[$, then $S_n(x)$ converges uniformly on $[a, b]$.
- If $u \in E$ is continuous and $u(-\pi) = u(\pi)$, then $\overline{S_n}(x)$ converges uniformly to $u(x)$.

The statement $u' \in E$ does not mean that $u'(x)$ exists everywhere, but that there exists a $v \in E$ such that $v(x) = u'(x)$ when $u'(x)$ exists and that u' exists everywhere except at a finite number of points in $[a, b]$.

2.4 General Fourier Series

- For a given ON system, the complex numbers $\langle v, \, e_k \rangle$, $k = 1, 2, \ldots$, are called the **generalized Fourier coefficients** of v.
- If $W = \{e_1, e_2, \ldots\}$ is an ON system in V, then W is closed if and only if **Parseval's identity** holds:

$$\sum_{k=1}^{\infty} |\langle v, \, e_k \rangle|^2 = \|v\|^2, \quad v \in V,$$

  or if $a_k = \langle u, \, e_k \rangle$ and $b_k = \langle v, \, e_k \rangle$, then

$$\langle u, \, v \rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}.$$

2.5 Rules for Fourier Coefficients

Let $u, v \in E$ be periodic with period $T > 0$ and define $\Omega = 2\pi/T$.

Table 1: Rules for Fourier Coefficients

Function	Fourier coefficient	Notes
$c_1 u(x) + c_2 v(x)$	$c_1 \widehat{u}[n] + c_2 \widehat{v}[n]$	
$(u * v)_T(x)$	$\widehat{u}[n]\,\widehat{v}[n]$	periodic convolution†
$u(x)v(x)$	$(\widehat{u} * \widehat{v})[n]$	
$e^{im\Omega x} u(x)$	$\widehat{u}[n-m]$	$m \in \mathbf{Z}$
$u(x-a)$	$e^{-in\Omega a}\,\widehat{u}[n]$	$a \in \mathbf{R}$
$u(ax)$	$\widehat{u}[n]$	period T/a, $a > 0$
$u(-x)$	$\widehat{u}[-n]$	
$\overline{u(x)}$	$\overline{\widehat{u}[-n]}$	
$u'(x)$	$in\Omega\widehat{u}[n]$	
$u^{(k)}(x)$	$(i\Omega n)^k \widehat{u}[n]$	$k = 1, 2, \ldots$

† $(u * v)_T(x) = \frac{1}{T}\int_0^T u(x-t)v(t)\,dt.$

3 The Fourier Transform

The **Fourier transform** of a function $u\colon \mathbf{R} \to \mathbf{C}$ given by

$$U(\omega) = \widehat{u}(\omega) = \mathcal{F}\,u(\omega) = \int_{-\infty}^{\infty} u(x)e^{-i\omega x}\,dx, \quad \omega \in \mathbf{R},$$

when this integral exists.

- If $u \in L^1(\mathbf{R})$ then $\mathcal{F}\,u(\omega)$ exists for all $\omega \in \mathbf{R}$ and

$$\|\,\mathcal{F}\,u\|_\infty \le \|u\|_{L^1(\mathbf{R})}.$$

- For $u \in G$, the Fourier transform $\mathcal{F}\,u$ is uniformly continuous on $\mathbf{R}$.
- **The Riemann-Lebesgue lemma**: For $u \in G$ we have $\mathcal{F}\,u(\omega) \to 0$ as $|w| \to \infty$.

3.1 Convergence

Kernels

- The **Dirichlet kernel** (on $\mathbf{R}$):

$$D_R(x) = \frac{\sin(Rx)}{\pi x}, \quad x \neq 0,$$

and $D_R(0) = R/\pi$.

- The **Fejér kernel** (on $\mathbf{R}$):

$$F_M(t) = \frac{1}{2\pi}\int_{-M}^{M}\left(1 - \frac{|\omega|}{M}\right)e^{i\omega t}\,d\omega = \frac{1-\cos Mx}{\pi M x^2} = \frac{M}{2\pi}\left(\frac{\sin(Mx/2)}{Mx/2}\right)^2,$$

where the last two equalities assumes that $x \neq 0$.

Inversion

- If $u \in G(\mathbf{R})$ and $D^{\pm}u(x)$ exists, then

$$\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R}\mathcal{F}\,u(\omega)e^{i\omega x}\,d\omega = \frac{u(x^+)+u(x^-)}{2}.$$

- If $u \in G(\mathbf{R})$, then

$$\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R}\mathcal{F}\,u(\omega)\left(1-\frac{|\omega|}{R}\right)e^{i\omega x}\,d\omega = \frac{u(x^+)+u(x^-)}{2}.$$

- If $u \in G(\mathbf{R})$, then

$$\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R}\mathcal{F}\,u(\omega)e^{i\omega x}\,d\omega = \frac{u(x^+)+u(x^-)}{2},$$

whenever the limit exists.

- **Uniqueness**: If $u, v \in G(\mathbf{R})$ and $\mathcal{F}\,u(\omega) = \mathcal{F}\,v(\omega)$ for every $\omega \in \mathbf{R}$, then $u(x) = v(x)$ for every $x \in \mathbf{R}$ where both u and v are continuous.

3.2 Special Rules

- If $u, U \in G(\mathbf{R})$ and $U(\omega) = \mathcal{F}(u)(\omega)$, then

$$\mathcal{F}^{-1}(U)(x) = \frac{1}{2\pi}\mathcal{F}\big((\mathcal{F}\,u)(-\omega)\big)(x) \quad \text{and} \quad \mathcal{F}(\mathcal{F}\,u(\omega))(x) = 2\pi u(-x),$$

for every x where u is continuous and $D^{\pm}u(x)$ exist.

- If $u, v \in G(\mathbf{R})$ such that $uv, \mathcal{F}\,u, \mathcal{F}\,v \in G(\mathbf{R})$, then

$$\mathcal{F}(uv)(\omega) = \frac{1}{2\pi}(\mathcal{F}(u) * \mathcal{F}(v))(\omega).$$

3.3 Plancherel's Theorem

- If $u \in G(\mathbf{R}) \cap L^2(\mathbf{R})$, then $\mathcal{F}\,u \in L^2(\mathbf{R})$.
- **Parseval's formula**: If $u, v \in G(\mathbf{R}) \cap L^2(\mathbf{R})$, then

$$\int_{-\infty}^{\infty} u(x)\,\overline{v(x)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}\,u(\omega)\,\overline{\mathcal{F}\,v(\omega)}\,d\omega.$$

3.4 Rules for the Fourier Transform

Let $u, v \in G(\mathbf{R})$ with $U(\omega) = \mathcal{F}\,u(\omega)$ and $V(\omega) = \mathcal{F}\,v(\omega)$.

Table 2: Rules for the Fourier transform

Function	Fourier transform	Notes
$c_1u(x)+c_2v(x)$	$c_1U(\omega)+c_2V(\omega)$	
$(u*v)(x)$	$U(\omega)V(\omega)$	
$e^{iax}u(x)$	$U(\omega-a)$	$a\in\mathbf{R}$
$u(x)\cos ax$	$\dfrac{U(\omega-a)+U(\omega+a)}{2}$	$a\in\mathbf{R}$
$u(x)\sin ax$	$\dfrac{U(\omega-a)-U(\omega+a)}{2i}$	$a\in\mathbf{R}$
$u(x-x_0)$	$e^{-ix_0\omega}U(\omega)$	$x_0\in\mathbf{R}$
$u(ax)$	$\dfrac{1}{\lvert a\rvert}U\left(\dfrac{\omega}{a}\right)$	$a\in\mathbf{R}$, $a\neq 0$
$\overline{u(x)}$	$\overline{U(-\omega)}$	
$u'(x)$	$i\omega U(\omega)$	$u\in C(\mathbf{R})$, $u'\in G$
$u^{(k)}(x)$	$(i\omega)^kU(\omega)$	$u^{(k)}\in G(\mathbf{R})$
$x^mu(x)$	$i^mU^{(m)}(\omega)$	$x^mu(x)\in G$, $m=1,2,3,\ldots$

3.5 Fourier Transforms

Table 3: Fourier transforms

Function	Fourier transform	Notes
e^{-ax^2}	$\sqrt{\dfrac{\pi}{a}}e^{-\omega^2/4a}$	$a>0$
$e^{-a\lvert x\rvert}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
$\operatorname{sgn}(x)e^{-a\lvert x\rvert}$	$\dfrac{-2i\omega}{a^2+\omega^2}$	$a>0$
$H(x)e^{-ax}$	$\dfrac{1}{a+i\omega}$	$\operatorname{Re} a>0$
$H(-x)e^{ax}$	$\dfrac{1}{a-i\omega}$	$\operatorname{Re} a>0$
$\dfrac{1}{a^2+x^2}$	$\dfrac{\pi}{a}e^{-a\lvert\omega\rvert}$	$a>0$
$H(x+a)-H(x-a)$	$\dfrac{2\sin a\omega}{\omega}$	$a>0$
$\operatorname{sgn}(x)\big(H(x+a)-H(x-a)\big)$	$\dfrac{2(1-\cos a\omega)}{i\omega}$	$a>0$
$(a-\lvert x\rvert)\big(H(x+a)-H(x-a)\big)$	$\dfrac{2(1-\cos a\omega)}{\omega^2}$	$a>0$
$\dfrac{1-\cos at}{t^2}$	$\pi(a-\lvert\omega\rvert)\big(H(\omega+a)-H(\omega-a)\big)$	$a>0$

4 The (unilateral) Laplace Transform

The Laplace transform of $u\colon [0,\infty[\to \mathbf{C}$ is given by

$$\mathcal{L}\,u(s)=\int_0^\infty u(t)e^{-st}\,dt,$$

for those $s=\sigma+i\omega\in\mathbf{C}$, $\sigma,\omega\in\mathbf{R}$, where this integral is convergent.

- **Exponential growth**: A piecewise continuous $u\colon [0,\infty[$ is of exponential growth (of order a) if there exists constants $a>0$ and $K>0$ such that $|u(t)|\le Ke^{at}$ for $t\ge 0$. The set of all such functions will be denoted by X_a.
- **Existence of $\mathcal{L}\,u(s)$**: If $u\in X_a$ for some $a>0$, then the Laplace transform $\mathcal{L}\,u(s)$ exists (at least) for $\operatorname{Re} s>a$.
- $\mathcal{L}\,u(s)\to 0$ as $\mathbf{R}\ni s\to\infty$.
- $\mathcal{L}\,u(s)$ converges uniformly for $\operatorname{Re} s>a$.
- $\mathcal{L}\,u(s)$ is analytic for $\operatorname{Re} s>a$.
- **Periodicity**: If there exists $T>0$ such that $u(t+T)=u(t)$ for every $t\ge 0$, then

$$\mathcal{L}\,u(s)=\frac{1}{1-e^{-sT}}\int_0^T u(\tau)e^{-s\tau}\,d\tau.$$

4.1 Inversion

- If $u \in X_a$ has right- and lefthand limits at a point $t > 0$, then

$$\lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \mathcal{L}\, u(\sigma + i\omega) e^{\sigma t} e^{i\omega t}\, d\omega = \frac{u(t^+) + u(t^-)}{2},$$

  where the vertical line $\operatorname{Re} z = \sigma$ is contained in the region of convergence of $\mathcal{L}\, u(s)$

- If $u, v \in X_a$ and $\mathcal{L}\, u(s) = \mathcal{L}\, v(s)$ on some vertical line $\operatorname{Re} s = \sigma$, then $u(t) = v(t)$ for all t where u and v are continuous.

4.2 Limit Theorems

- **Final value theorem**:
  If $u\colon [0, \infty[\to \mathbf{C}$ is bounded and $\lim_{t\to\infty} u(t) = A$, then $A = \lim_{\mathbf{R}\ni s\to 0^+} s\, \mathcal{L}\, u(s)$.

- **Initial value theorem**:
  If $u\colon [0, \infty[\to \mathbf{C}$ belongs to X_b and $\lim_{t\to 0^+} u(t) = a$, then $a = \lim_{\mathbf{R}\ni s\to\infty} s\, \mathcal{L}\, u(s)$.

4.3 Rules for the Laplace Transform

Let $U(s) = \mathcal{L}\, u(t)$, $\sigma > \sigma_u$ and $V(s) = \mathcal{L}\, v(t)$, $\sigma > \sigma_v$.

Table 4: Rules for Laplace transforms

Function	Unilateral Laplace transform	Region of convergence
$c_1 u(t) + c_2 v(t)$	$c_1 U(s) + c_2 V(s)$	$\sigma > \max\{\sigma_u, \sigma_v\}$
$(u * v)(t)$	$U(s)V(s)$	unilateral conv.†; $\sigma > \max\{\sigma_u, \sigma_v\}$
$e^{at} u(t)$	$U(s - a)$	$\sigma > \sigma_u + \operatorname{Re} a$
$u(t - t_0) H(t - t_0)$	$e^{-t_0 s} U(s)$	$\sigma > \sigma_u$, $t_0 > 0$
$u(at)$	$\frac{1}{a} U\left(\frac{s}{a}\right)$	$\sigma > a\sigma_u$, $a > 0$
$\overline{u(t)}$	$\overline{U(\overline{s})}$	$\sigma > \sigma_u$
$u'(t)$	$sU(s) - u(0)$	$\sigma > \sigma_u$
$u''(t)$	$s^2 U(s) - su(0) - u'(0)$	$\sigma > \sigma_u$
$u^{(n)}(t)$	$s^n U(s) - s^{n-1} u(0) - \cdots$ $\cdots - su^{(n-2)}(0) - u^{(n-1)}(0)$	$\sigma > \max\{\sigma_u, \sigma_{u'}, \ldots, \sigma_{u^{(n-1)}}\}$
$\int_0^t u(\tau)\, d\tau$	$\frac{U(s)}{s}$	$\sigma > \max\{\sigma_u, 0\}$
$t^m u(t)$	$(-1)^m U^{(m)}(s)$	$\sigma > \sigma_u$

† $(u * v)(t) = \int_0^t u(\tau) v(t - \tau)\, d\tau$.

4.4 Laplace Transforms

Table 5: Laplace transforms

Function	Unilateral Laplace transform	Region of convergence
$H(t) = 1$	$\frac{1}{s}$	$\sigma > 0$
t	$\frac{1}{s^2}$	$\sigma > 0$
t^m	$\frac{m!}{s^{m+1}}$	$\sigma > 0$, $m = 1, 2, 3, \ldots$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$	$\sigma > 0$, $a > 0$
e^{at}	$\frac{1}{s - a}$	$\sigma > \operatorname{Re} a$
$t^m e^{at}$	$\frac{m!}{(s-a)^{m+1}}$	$\sigma > \operatorname{Re} a$, $m = 1, 2, 3, \ldots$
$\cos at$	$\frac{s}{s^2 + a^2}$	$\sigma > \lvert \operatorname{Im} a \rvert$
$t \cos at$	$\frac{s^2 - a^2}{(s^2 + a^2)^2}$	$\sigma > \lvert \operatorname{Im} a \rvert$
$\sin at$	$\frac{a}{s^2 + a^2}$	$\sigma > \lvert \operatorname{Im} a \rvert$
$t \sin at$	$\frac{2as}{(s^2 + a^2)^2}$	$\sigma > \lvert \operatorname{Im} a \rvert$
$\frac{\sin at}{t}$	$\arctan\left(\frac{a}{s}\right)$	$\sigma > \lvert \operatorname{Im} a \rvert$
$\cosh at$	$\frac{s}{s^2 - a^2}$	$\sigma > \lvert \operatorname{Re} a \rvert$
$\sinh at$	$\frac{a}{s^2 - a^2}$	$\sigma > \lvert \operatorname{Re} a \rvert$
$J_0(at)$	$\frac{1}{\sqrt{a^2 + s^2}}$	$\sigma > \lvert \operatorname{Im} a \rvert$

5 The (unilateral) Z Transform

The **Z transform** of a sequence $u[k]$, $k = 0, 1, 2, \ldots$, is defined by

$$\mathcal{Z}(u)(z) = \sum_{k=0}^{\infty} u[k]\, z^{-k},$$

for those $z = x + iy \in \mathbf{C}$, $x, y \in \mathbf{R}$, where this series is absolutely convergent.

- **Existence of $\mathcal{Z}\,u(z)$**: For a sequence $u[k]$, $k = 0, 1, 2, \ldots$, the Z transform $\mathcal{Z}\,u(z)$ has a region of convergence R such that $\mathcal{Z}\,u(z)$ is absolutely (uniformly) convergent for $|z| > R$ and divergent for $|z| < R$. It is possible that $R = 0$ or $R = \infty$.

- **Inversion**: If $U(z) = \mathcal{Z}\,u(z)$, then

$$u[k] = \frac{1}{2\pi i} \oint_\gamma z^{k-1} U(z)\, dz, \quad k = 0, 1, 2, \ldots,$$

where γ is a closed curve (inside $|z| > R_u$) counterclockwise around the origin.

- **Uniqueness**: If $\mathcal{Z}\,u(z) = \mathcal{Z}\,v(z)$ for all $|z| > R$ for some $R > 0$, then $u[k] = v[k]$ for $k = 0, 1, 2, \ldots$.

- **Initial value theorem**: If there's an $R > 0$ such that $\mathcal{Z}\,u(z)$ exists for $|z| > R$, then

$$\lim_{|z|\to\infty} \mathcal{Z}\,u(z) = u[0].$$

5.1 Rules for the Z Transform

Let $U(z) = \mathcal{Z}(u[k])(z)$, $|z| > R_u$ and $V(z) = \mathcal{Z}(v[k])(z)$, $|z| > R_v$.

Table 6: Rules for Z transforms

Function	Unilateral Z transform transform	Region of convergence
$c_1 u[k] + c_2 v[k]$	$c_1 U(z) + c_2 V(z)$	$\|z\| > \max\{R_u, R_v\}$
$(u * v)[k]$	$U(z)V(z)$	unilateral conv.†; $\|z\| > \max\{R_u, R_v\}$
$a^k u[k]$	$U\left(\frac{z}{a}\right)$	$\|z\| > \|a\| R_u$, $a \in \mathbf{C} \setminus \{0\}$
$u[k-m]H[k-m]$	$z^{-m}U(z)$	$\|z\| > R_u$, $m = 1, 2, 3, \ldots$
$u[k-m]$	$z^{-m}U(z) + z^{-m+1}u[-1] + \cdots \cdots + z^{-1}u[-m+1] + u[-m]$	$\|z\| > R_u$, $m = 1, 2, 3, \ldots$
$u[k+m]$	$z^m U(z) - z^m u[0] + \cdots \cdots - z^2 u[m-2] - z u[m-1]$	$\|z\| > R_u$, $m = 1, 2, 3, \ldots$
$\overline{u[k]}$	$\overline{U(\overline{z})}$	$\|z\| > R_u$
$\sum_{l=0}^{k} u[l]$	$\frac{z}{z-1} U(z)$	$\|z\| > \max\{R_u, 1\}$
$k^m u[k]$	$\left(-z\frac{d}{dz}\right)^m U(z)$	$\|z\| > R_u$

† $(u * v)[k] = \sum_{l=0}^{k} u[l]v[k-l]$.

5.2 Z Transforms

Table 7: Z transforms

Function	Unilateral Z transform	Region of convergence
$\delta[k]$	1	$z \in \mathbf{C}$
$\delta[k-m]$	z^{-m}	$\|z\| > 0$, $m = 1, 2, \ldots$
$H[k]$	$\frac{z}{z-1}$	$\|z\| > 1$
k	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
a^k	$\frac{z}{z-a}$	$\|z\| > \|a\|$
ka^k	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
$k^2 a^k$	$\frac{az^2 + a^2 z}{(z-a)^3}$	$\|z\| > \|a\|$
$k^3 a^k$	$\frac{az^3 + 4a^2 z^2 + a^3 z}{(z-a)^4}$	$\|z\| > \|a\|$
$(k+1)a^k$	$\frac{z^2}{(z-a)^2}$	$\|z\| > \|a\|$
$\binom{k+m}{m} a^k$	$\frac{z^{m+1}}{(z-a)^{m+1}}$	$\|z\| > \|a\|$, $m = 2, 3, \ldots$
$\binom{k}{m} a^k$	$\frac{a^m z}{(z-a)^{m+1}}$	$\|z\| > \|a\|$, $m = 2, 3, \ldots$
$\binom{k+n}{m} a^k$	$\frac{a^{m-n} z^{n+1}}{(z-a)^{m+1}}$	$\|z\| > \|a\|$, $m = 2, 3, \ldots$, $n = 1, \ldots, m-1$
$\cos k\alpha$	$\frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1}$	$\|z\| > 1$
$\sin k\alpha$	$\frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$	$\|z\| > 1$
$k\cos k\alpha$	$\frac{z^3\cos\alpha - 2z^2 + z\cos\alpha}{(z^2 - 2z\cos\alpha + 1)^2}$	$\|z\| > 1$
$k\sin k\alpha$	$\frac{z^3\sin\alpha - z\sin\alpha}{(z^2 - 2z\cos\alpha + 1)^2}$	$\|z\| > 1$
$\frac{a^k}{k!}$	$e^{a/z}$	$\|z\| > 0$
$\frac{1}{k}H[k-1]$	$\ln\frac{z}{z-1}$	$\|z\| > 1$
$\binom{n}{k} a^k b^{n-k}$	$\frac{(bz+a)^n}{z^n}$	$\|z\| > 0$, $n = 1, 2, 3, \ldots$