Transform theory 2022-06-03 – Solutions

1. (a) No. The function is discontinuous at (and unbounded near) $\omega = \pm 1$.
   (b) Yes, from the table we find that $f(t) = \frac{\pi^{-1}}{1+\omega^2}$ has this transform (and is a function from $G(\mathbf{R})$).
   (c) No. For example $f(x) = \frac{1}{\sqrt{|x|}}$, $0 < |x| < 1$, and $f(x) = 0$ elsewhere, is a function such that $f \in L^1(\mathbf{R})$ but $f \notin L^2(\mathbf{R})$.
   (d) We note that $\dfrac{s}{s-1} = 1 + \dfrac{1}{s-1} \to 1$ as $|s| \to \infty$. Therefore this can't be the Laplace transform of an exponentially bounded function since the Laplace transform must tend to zero as $|s| \to \infty$.
   (e) No. The fact that the limit of the integrals equals the integral of the limiting function does not imply that the convergence is uniform; See exercise 2.4.

   Answer: No, Yes, No, No, No.

2. (a) Let $g(t) = (3-t)H(3-t)$, $t \geq 0$. We then calculate $G(s)$ directly by using the definition:

$$G(s) = \int_0^3 (3-t)e^{-st}\,dt = \left[\frac{(3-t)e^{-st}}{-s}\right]_0^3 - \frac{1}{s}\int_0^3 e^{-st}\,dt = \frac{3}{s} - \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^3$$
$$= \frac{3}{s} + \frac{1}{s^2}\left(e^{-3s} - 1\right),$$

   where we assume that $\operatorname{Re} s > 0$.

   (b) We're looking for exponentially bounded solutions, so we assume that $|y(t)| \leq Ke^{at}$ for some constants $a, C > 0$. Taking the Laplace transform, we obtain that

$$s^2 Y(s) - sy(0) - y'(0) + 4Y(s) = 4G(s) \quad \Leftrightarrow \quad Y(s)\left(s^2+4\right) = 1 + 4G(s),$$

   where $4G(s)$ is the Laplace transform of the right-hand side (see previous exercise). Hence,

$$Y(s) = \frac{1}{s^2+4} + \frac{12}{s(s^2+4)} + \frac{4}{s^2(s^2+4)}\left(e^{-3s} - 1\right),$$

   Decomposition in partial fractions yields

$$\frac{4}{s^2(s^2+4)} = \frac{1}{s^2} - \frac{1}{s^2+4}$$

   which also gives a suitable decomposition for the the middle fraction:

$$\frac{12}{s(s^2+4)} = 3s \cdot \frac{4}{s^2(s^2+4)} = \frac{3}{s} - \frac{3s}{s^2+4}.$$

   From a table,

$$\mathcal{L}(\cos 2t) = \frac{s}{s^2+4}, \quad \mathcal{L}(\sin 2t) = \frac{2}{s^2+4}, \quad \mathcal{L}(t) = s^{-2}, \quad \mathcal{L}(1) = s^{-1},$$
$$\mathcal{L}(H(t-t_0)u(t-t_0)) = U(s)e^{-st_0},$$

so

$$\begin{aligned} y(t) &= \frac{\sin 2t}{2} + 3 - 3\cos 2t - t + \frac{\sin 2t}{2} + H(t-3)\left(t - 3 - \frac{\sin 2(t-3)}{2}\right) \\ &= \sin 2t - 3\cos 2t + 3 - t + H(t-3)\left(t - 3 - \frac{\sin 2(t-3)}{2}\right) \\ &= \begin{cases} \sin 2t - 3\cos 2t + 3 - t, & 0 \le t < 3, \\ \sin 2t - 3\cos 2t - \frac{1}{2}\sin 2(t-3), & t \ge 3, \end{cases} \end{aligned}$$

by uniqueness.

Answer: $y(t) = \sin 2t - 3\cos 2t + 3 - t + H(t-3)\left(t - 3 - \dfrac{\sin 2(t-3)}{2}\right), \quad t \ge 0.$

3. Clearly $u \in E$. We find that, for $k \neq 0$,

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx = \frac{1}{2\pi}\int_0^{\pi} xe^{-ikx}\,dx = \frac{1}{2\pi}\left(\left[x\frac{e^{-ikx}}{-ik}\right]_0^{\pi} + \frac{1}{ik}\int_0^{\pi} e^{-ikx}\,dx\right) \\ &= \frac{1}{2\pi}\left(\frac{\pi(-1)^k}{-ik} + \frac{1}{k^2}\left[e^{-ikx}\right]_0^{\pi}\right) = \frac{i(-1)^k}{2k} + \frac{(-1)^k - 1}{2\pi k^2} \end{aligned}$$

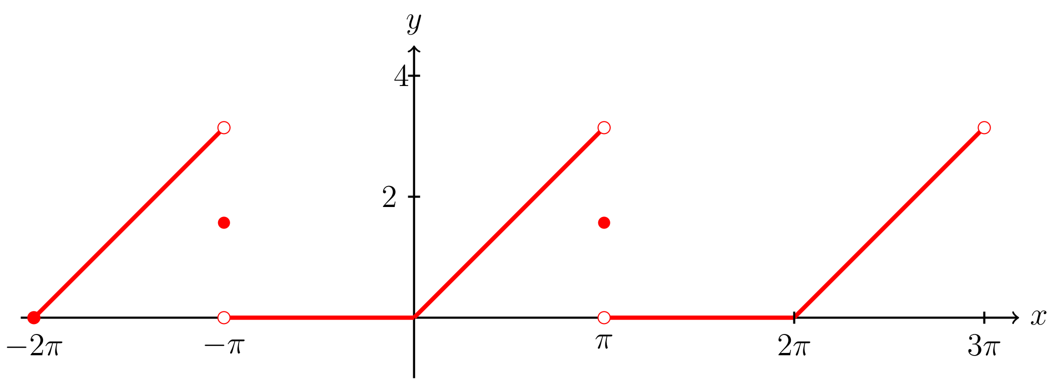
and

$$c_0 = \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{\pi^2}{4\pi} = \frac{\pi}{4}.$$

Hence

$$u(x) \sim \frac{\pi}{4} + \sum_{k\neq 0}\left(\frac{i(-1)^k}{2k} + \frac{(-1)^k - 1}{2\pi k^2}\right) e^{ikx}.$$

Furthermore, the function is differentiable for $x \neq m\pi$, $m \in \mathbf{Z}$, and the function has right- and lefthand derivatives at $x = m\pi$ (it is piecewise straight lines). Hence – by Dirichlet's theorem – the Fourier series is convergent and converges to $u(x)$ for every $x \neq m\pi$, to 0 at $x = 2l\pi$ ($l \in \mathbf{Z}$) and to $\pi/2$ for $x = (2l+1)\pi$ ($l \in \mathbf{Z}$). Due to the discontinuity at, e.g., $x = \pi$, the convergence can not be uniform on $[-\pi, \pi]$.



Answer: $u(x) \sim \dfrac{\pi}{4} + \displaystyle\sum_{k\neq 0}\left(\frac{i(-1)^k}{2k} + \frac{(-1)^k - 1}{2\pi k^2}\right) e^{ikx}$; see above.

4. The sum in the equation is the convolution of u with the function $k \mapsto 2^k$. We take the Z transform of the equation and find that

$$zU(z) - zu[0] - 2U(z) + 4U(z)\frac{z}{z-2} = 0,$$

where we assume that (at least) $|z| > 2$. Reformulating this equation, we find that

$$U(z)\left(z - 2 + \frac{4z}{z-2}\right) = z \quad \Leftrightarrow \quad U(z)\frac{z^2+4}{z-2} = z$$
$$\Leftrightarrow \quad U(z) = \frac{z(z-2)}{z^2+4} = \frac{z^2}{z^2+4} - \frac{2z}{z^2+4}.$$

We note that

$$\frac{z^2}{z^2+4} = \frac{4 \cdot (z/2)^2}{4((z/2)^2+1)} = \frac{(z/2)^2}{(z/2)^2+1}$$

and since $\mathcal{Z}(\cos\frac{n\pi}{2}) = \frac{z^2}{z^2+1}$, we find that

$$\mathcal{Z}\left(2^n \cos\frac{n\pi}{2}\right) = \frac{z^2}{z^2+4}.$$

Similarly,

$$\frac{2z}{z^2+4} = \frac{4 \cdot (z/2)}{4((z/2)^2+1)} = \frac{z/2}{(z/2)^2+1}$$

and since $\mathcal{Z}(\sin\frac{n\pi}{2}) = \frac{z}{z^2+1}$, we find that

$$\mathcal{Z}\left(2^n \sin\frac{n\pi}{2}\right) = \frac{2z}{z^2+4}.$$

By uniqueness,

$$u[n] = 2^n \cos\frac{n\pi}{2} - 2^n \sin\frac{n\pi}{2} = 2^n\left(\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right).$$

Answer: $2^n\left(\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right)$, $n = 0, 1, 2, 3, \ldots$.

5. We note that $f(t) = (1-\cos 4t)/t^2$, $t \neq 0$, is a function that belongs to $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. This is clear since $\cos 4t = 1 - (4t)^2/2 + O(t^4)$ close to zero, so we have no problems at the origin (f is continuous everywhere). When $|t|$ is large,

$$\left|\frac{1-\cos 4t}{t^2}\right|^2 \leq \frac{4}{t^4},$$

which is clearly integrable for large t. From a table, we find that

$$\mathcal{F}(f)(\omega) = \begin{cases} \pi(4-|\omega|), & |\omega| < 4, \\ 0, & |\omega| \geq 4. \end{cases}$$

By Parseval's identity, we can conclude that

$$\int_{-\infty}^{\infty} \left(\frac{1-\cos 4t}{t^2}\right)^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{F}(f)(\omega)|^2 \, d\omega = \frac{1}{2\pi}\int_{-4}^{4} (\pi(4-|\omega|))^2 \, d\omega$$
$$= \frac{\pi}{2}\cdot 2\int_0^4 (4-\omega)^2 \, d\omega = \pi\left[-\frac{(4-\omega)^3}{3}\right]_0^4 = \frac{4^3\pi}{3} = \frac{64\pi}{3}.$$

Answer: $\frac{64\pi}{3}$.

6. (a) If $u \in L^1(\mathbf{R})$ then the integral defining the Fourier transform will given as an absolutely convergent integral since $|u(x)e^{-i\omega x}| \leq |u(x)|$ (for ω real) and

$$|\,\mathcal{F}\,u(\omega)| = \left|\int_{-\infty}^{\infty} u(x)\,e^{-i\omega x}\,dx\right| \leq \int_{-\infty}^{\infty} |u(x)||e^{-i\omega x}|\,dx = \int_{-\infty}^{\infty} |u(x)|\,dx < \infty,$$

independently of ω. Hence $\mathcal{F}\,u$ is bounded at every point.

(b) Using integration by parts, we see that

$$\int_{-M}^{M} u'(x)e^{-i\omega x}\,dx = /\text{ I.B.P. }/ = u(M)e^{-i\omega M} - u(-M)e^{i\omega M} + i\omega\int_{-M}^{M} u(x)e^{-i\omega x}\,dx$$
$$\to i\omega\int_{-\infty}^{\infty} u(x)e^{-i\omega x}\,dx = i\omega\,\mathcal{F}\,u(\omega), \text{ as } M \to \infty,$$

since $u(x) \to 0$ as $x \to \pm\infty$ ($e^{\pm i\omega M}$ is bounded).

Answer: Se above.

7. The series is uniformly convergent by the Weierstrass M-test since

$$\left|\frac{\arctan k}{k^2(k+1)}\cos\frac{kx}{2}\right| \leq \frac{\pi/2}{k^2(k+1)} \qquad \text{and} \qquad \sum_{k=1}^{\infty}\frac{1}{k^3} \text{ is convergent.}$$

Since we're summing continuous functions, the uniform convergence implies that $u(x)$ is a continuous function. Furthermore, looking at the derivatives of the functions we're summing, we find that

$$\left|\frac{d}{dx}\left(\frac{\arctan k}{k^2(k+1)}\cos\frac{kx}{2}\right)\right| = \left|-\frac{k\arctan k}{2k^2(k+1)}\sin\frac{kx}{2}\right| \leq \frac{\pi}{4k(k+1)} \quad \text{and since} \quad \sum_{k=1}^{\infty}\frac{1}{k^2} < \infty,$$

it follows by Weierstrass M-test that $\sum_{k=1}^{\infty} u_k'(x)$ converges uniformly. Furthermore, each derivative is a continuous function. These facts imply that $u \in C^1$. Hence, by Dirichlet's theorem ($u \in C^1$ implies that one-sided derivatives exist), we know that the Fourier series for u converges on $[-\pi,\pi]$. Since u is continuous on $[-\pi,\pi]$ and obviously $u(-\pi) = u(\pi$, this implies that the Fourier series converges to $u(x)$ for $-\pi \leq x \leq \pi$.

Now to find Fourier coefficients, we use the uniform convergence of the series $u(x)$ to calculate

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=1}^{\infty}\frac{\arctan k}{k^2(k+1)}\cos\frac{kx}{2}\right)e^{-imx}\,dx = \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{\arctan k}{k^2(k+1)}\int_{-\pi}^{\pi}\cos\frac{kx}{2}e^{-imx}\,dx.$$

If $k \neq 2|m|$, then

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos\frac{kx}{2} e^{-imx}\,dx &= \int_{-\pi}^{\pi} \left(\frac{e^{ikx/2}+e^{-ikx/2}}{2}\right) e^{-imx} dx = \frac{1}{2}\int_{-\pi}^{\pi} \left(e^{i(k-2m)x/2} + e^{-i(k+2m)x/2}\right) dx \\
&= \frac{1}{2}\left[\frac{2e^{i(k-2m)x/2}}{i(k-2m)} - \frac{2e^{-i(k+2m)x/2}}{i(k+2m)}\right]_{-\pi}^{\pi} = \left[\frac{e^{i(k-2m)x/2}}{i(k-2m)} - \frac{e^{-i(k+2m)x/2}}{i(k+2m)}\right]_{-\pi}^{\pi} \\
&= \frac{e^{i(k-2m)\pi/2}}{i(k-2m)} - \frac{e^{-i(k+2m)\pi/2}}{i(k+2m)} - \frac{e^{-i(k-2m)\pi/2}}{i(k-2m)} + \frac{e^{i(k+2m)\pi/2}}{i(k+2m)} \\
&= \frac{2\sin((k-2m)\pi/2)}{k-2m} + \frac{2\sin((k+2m)\pi/2)}{k+2m} \\
&= \frac{2(-1)^m\sin(k\pi/2)}{k-2m} + \frac{2(-1)^m\sin(k\pi/2)}{k+2m} \\
&= \frac{2(k+2m+k-2m)(-1)^m\sin(k\pi/2)}{(k-2m)(k+2m)} = \frac{(-1)^m 4k\sin(k\pi/2)}{k^2-4m^2}
\end{aligned}$$

and if $k = 2|m|$, then

$$\int_{-\pi}^{\pi} \cos\frac{2|m|x}{2} e^{-imx}\,dx = \pi$$

since $\sin(2|m|\pi/2) = \sin(|m|\pi) = 0$ (where we reused some calculations from above). The term π comes from either $e^{i(k-2m)x/2}$ or $e^{i(k+2m)x/2}$, depending on if $k = 2m$ or if $k = -2m$. Notice also that if k is even and $k \neq 2|m|$, then

$$\int_{-\pi}^{\pi} \cos\frac{kx}{2} e^{-imx}\,dx = 0$$

since $\sin(k\pi/2) = 0$ in this case. Therefore it is sufficient to sum when k is odd (except for when $k = 2|m|$). Thus, for $|m| \geq 1$,

$$\begin{aligned}
c_m &= \frac{\pi}{2\pi}\cdot\frac{\arctan(2|m|)}{4m^2(2|m|+1)} + \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{\arctan(2k+1)}{(2k+1)^2(2k+2)}\frac{(-1)^m 4(2k+1)(-1)^k}{(2k+1)^2-4m^2} \\
&= \frac{\arctan(2|m|)}{8m^2(2|m|+1)} + \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+m}\arctan(2k+1)}{(2k+1)(k+1)((2k+1)^2-4m^2)},
\end{aligned}$$

where the term outside the series appears due to the term for which $k = 2|m|$. Moreover, since

$$\int_{-\pi}^{\pi} \cos(kx/2)\,dx = \frac{4\sin(k\pi/2)}{k},$$

we have

$$\begin{aligned}
c_0 &= \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{\arctan k}{k^2(k+1)}\frac{4\sin(k\pi/2)}{k} = \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{\arctan(2k+1)}{(2k+1)^2(2k+2)}\frac{4(-1)^k}{2k+1} \\
&= \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k\arctan(2k+1)}{(2k+1)^3(k+1)}.
\end{aligned}$$

If that wasn't the most beautiful Fourier coefficients, I don't know what would be...

Answer:

$$c_0 = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \arctan(2k+1)}{(2k+1)^3(k+1)},$$

$$c_m = \frac{\arctan(2|m|)}{8m^2(2|m|+1)} + \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+m} \arctan(2k+1)}{(2k+1)(k+1)((2k+1)^2 - 4m^2)}, \quad |m| \geq 1.$$