Transform theory 2022-08-18 – Solutions

1. (a) No. The function does not tend to zero as $|\omega| \to \infty$ so the Riemann-Lebesgue lemma is not satisfied.

   (b) No. For example $f(x) = \frac{1}{x}$, $x > 1$, and $f(x) = 0$ elsewhere, is a function such that $f \in L^2(\mathbf{R})$ but $f \notin L^1(\mathbf{R})$.

   (c) Yes. A continuous funktion is bounded on a closed interval, so let K be such that $|u(x)| \leq K$ on $[-\pi, \pi]$. Then

   $$\int_{-\pi}^{\pi} |u(x)|\, dx \leq K \int_{-\pi}^{\pi} dx = 2\pi K < \infty$$

   and

   $$\int_{-\pi}^{\pi} |u(x)|^2\, dx \leq K^2 \int_{-\pi}^{\pi} dx = 2\pi K^2 < \infty,$$

   so $u \in L^1(-\pi, \pi)$ and $u \in L^2(-\pi, \pi)$.

   (d) Considering that $u(x) = (1+x^2)^{-1}$ is a C^1-funktion on $]-\pi, \pi[$, continuous on $[-\pi, \pi]$ with $u(-\pi) = u(\pi)$, it follows from a well known theorem that the convergence of the Fourier series is uniform.

   (e) No. For instance, $u[k] = 1$ has the Z transform $z/(z-1)$ for $|z| > 1$ and clearly this expression is unbounded as $z \to 1$.

   Answer: No, No, Yes, Yes, No.

2. We take the Z transform of the equation and find that

   $$z^2 U(z) - z^2 u[0] - z u[1] + z U(z) - z u[0] - 2U(z) = \frac{z}{z-1},$$

   where we assume that (at least) $|z| > 1$. Reformulating this equation, we find that

   $$U(z)\left(z^2 + z - 2\right) = 3z^2 + \frac{10z}{3} + \frac{z}{z-1} \quad \Leftrightarrow \quad \frac{U(z)}{z} = \frac{3z + 10/3}{(z-1)(z+2)} + \frac{1}{(z-1)^2(z+2)}$$

   $$\Leftrightarrow \quad \frac{U(z)}{z} = \frac{19/9}{z-1} + \frac{8/9}{z+2} + \frac{1/3}{(z-1)^2} - \frac{1/9}{z-1} + \frac{1/9}{z+2} = \frac{1/3}{(z-1)^2} + \frac{1}{z+2} + \frac{2}{z-1},$$

   where we assume that $|z| > 2$. From a table we find that

   $$\mathcal{Z}\,(k) = \frac{z}{(z-1)^2}, \quad \mathcal{Z}\left((-2)^k\right) = \frac{z}{z+2}, \quad \text{and} \quad \mathcal{Z}\,(1) = \frac{z}{z-1},$$

   so linearity and uniqueness imply that

   $$u[k] = \frac{k}{3} + (-2)^k + 2.$$

   Answer: $u[k] = \dfrac{k}{3} + (-2)^k + 2$, $k = 0, 1, 2, 3, \ldots$

3. Clearly $u \in E$. We find that, for $k \neq 0$,

$$
\begin{aligned}
c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx = \frac{1}{2\pi}\int_0^{\pi}(\pi - x)e^{-ikx}\,dx \\
&= \frac{1}{2\pi}\left(\left[(\pi - x)\frac{e^{-ikx}}{-ik}\right]_0^{\pi} - \frac{1}{ik}\int_0^{\pi} e^{-ikx}\,dx\right) \\
&= \frac{1}{2\pi}\left(\frac{\pi}{ik} - \frac{1}{k^2}\left[e^{-ikx}\right]_0^{\pi}\right) = \frac{-i}{2k} + \frac{1-(-1)^k}{2\pi k^2} = \frac{1 - ik\pi - (-1)^k}{2\pi k^2}
\end{aligned}
$$

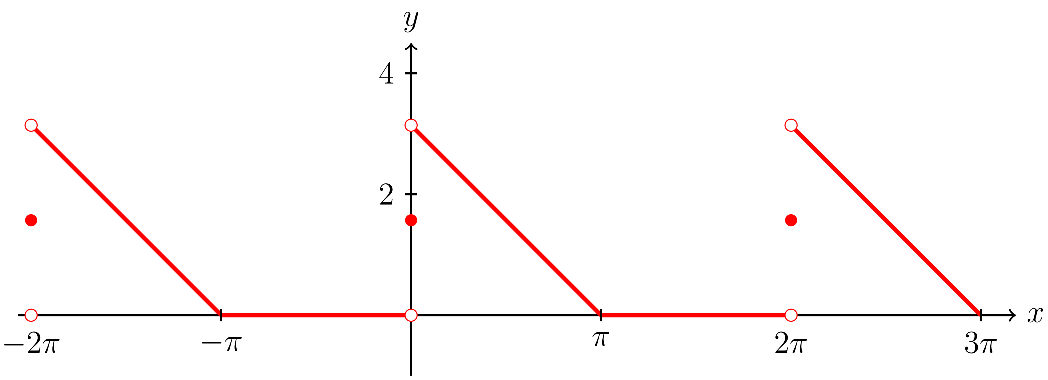
and

$$c_0 = \frac{1}{2\pi}\int_0^{\pi}(\pi - x)\,dx = \frac{\pi}{4}.$$

Hence

$$u(x) \sim \frac{\pi}{4} + \sum_{k\neq 0}\left(\frac{1 - ik\pi - (-1)^k}{2\pi k^2}\right) e^{ikx}.$$

Furthermore, the function u is differentiable for $x \neq m\pi$, $m \in \mathbf{Z}$, and the function has right- and lefthand derivatives at $x = m\pi$ (it is piecewise straight lines). Hence – by Dirichlet's theorem – the Fourier series is convergent and converges to $u(x)$ for every $x \neq m\pi$, to 0 at $x = (2l+1)\pi$ ($l \in \mathbf{Z}$) and to $\pi/2$ for $x = 2l\pi$ ($l \in \mathbf{Z}$).



For $k \neq 0$, note that

$$c_k = \frac{1 - ik\pi - (-1)^k}{2\pi k^2} = \frac{1-(-1)^k}{2\pi k^2} - i\frac{1}{2k}.$$

With $c_0 = \pi/4$, we therefore have

$$|c_k|^2 = \frac{(1-(-1)^k)^2}{4\pi^2 k^4} + \frac{1}{4k^2}, \quad k \neq 0, \quad |c_0|^2 = \frac{\pi^2}{16},$$

so

$$\sum_{k\in\mathbf{Z}} |c_k|^2 = \frac{\pi^2}{16} + 2\sum_{k=1}^{\infty}\frac{(1-(-1)^k)^2}{4\pi^2 k^4} + \frac{1}{4k^2} = \frac{\pi^2}{16} + 2\sum_{k=1}^{\infty}\frac{(1-(-1)^k)^2 + \pi^2 k^2}{4\pi^2 k^4}$$

since $|c_{-k}|^2 = |c_k|^2$. By Parseval's identity,

$$\sum_{k\in\mathbf{Z}} |c_k|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|u(x)|^2\,dx = \frac{1}{2\pi}\int_0^{\pi}(\pi - x)^2\,dx = \frac{\pi^2}{6}.$$

Rearrenging the terms, we obtain

$$2\sum_{k=1}^{\infty}\frac{(1-(-1)^k)^2+\pi^2k^2}{4\pi^2k^4}=\frac{\pi^2}{6}-\frac{\pi^2}{16}=\frac{5\pi^2}{48},$$

so

$$\sum_{k=1}^{\infty}\frac{(1-(-1)^k)^2+\pi^2k^2}{k^4}=\frac{5\pi^4}{24},$$

which was what we set out to prove.

Answer: $u(x)\sim\frac{\pi}{4}+\sum_{k\neq 0}\left(\frac{1-ik\pi-(-1)^k}{2\pi k^2}\right)e^{ikx}$; see above.

4. Assuming that $u,u',u''\in G$, we take the Fourier transform to find that

$$(i\omega)^2U(\omega)-i\omega U(\omega)-6U(\omega)=\mathcal{F}(25e^{-2x}H(x))=\frac{25}{2+i\omega}.$$

With $s=i\omega$, we see that $s^2-s-6=(s-3)(s+2)$, so

$$U(\omega)=\frac{25}{(2+i\omega)^2(i\omega-3)}=-\frac{5}{(2+i\omega)^2}-\frac{1}{3-i\omega}-\frac{1}{2+i\omega}.$$

Notice that

$$\frac{d}{d\omega}\left(\frac{1}{2+i\omega}\right)=-i\frac{1}{(2+i\omega)^2}$$

so

$$\frac{1}{(2+i\omega)^2}=i\frac{d}{d\omega}\left(\frac{1}{(1+i\omega)^2}\right)=\mathcal{F}\left(xe^{-2x}H(x)\right).$$

From a table, we also find that

$$\mathcal{F}(e^{-2x}H(x))=\frac{1}{2+i\omega}\quad\text{and}\quad\mathcal{F}(e^{3x}H(-x))=\frac{1}{3-i\omega},$$

so by uniqueness and linearity,

$$\begin{aligned}y(x)&=-5xe^{-x}H(x)-e^{3x}H(-x)-e^{-2x}H(x)=-(5x+1)e^{-2x}H(x)-e^{3x}H(-x)\\&=\begin{cases}-e^{-3x}, & x<0.\\-(5x+1)e^{-2x}, & x\geq 0.\end{cases}\end{aligned}$$

This function and its derivatives up to order 2 are absolutely integrable.

Answer: $y(x)=-(5x+1)e^{-2x}H(x)-e^{3x}H(-x)$.

5. We're looking for exponentially bounded solutions, so we assume that $|u(t)|\leq Ke^{at}$ for some constants $a,C>0$. Taking the Laplace transform of the equation, we obtain that

$$sU(s)-u(0)+U(s)+\frac{1}{s}U(s)=\int_0^1 e^{-st}\,dt=\frac{1-e^{-s}}{s}$$

so

$$U(s)\left(s+1+\frac{1}{s}\right)=U(s)\frac{s^2+s+1}{s}=1+\frac{1-e^{-s}}{s}.$$

Hence,

$$U(s) = \frac{s}{s^2+s+1} + \frac{1-e^{-s}}{s^2+s+1}.$$

Since

$$s^2+s+1 = (s+1/2)^2 + 3/4,$$

we rewrite

$$\frac{s}{s^2+s+1} = \frac{s+1/2}{(s+1/2)^2+3/4} - \frac{1/2}{(s+1/2)^2+3/4}$$

and

$$\frac{1}{s^2+s+1} = \frac{2}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+3/4}.$$

From a table,

$$\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}, \quad \mathcal{L}(\sin at) = \frac{a}{s^2+a^2}, \quad \mathcal{L}(e^{-at}f(t)) = F(s+a),$$
$$\mathcal{L}(H(t-t_0)f(t-t_0)) = F(s)e^{-st_0},$$

so

$$\begin{aligned}\frac{s+1/2}{(s+1/2)^2+3/4} - \frac{1/2}{(s+1/2)^2+3/4} &= \mathcal{L}\left(e^{-t/2}\cos\frac{t\sqrt{3}}{2}\right) - \frac{1}{2}\frac{2}{\sqrt{3}}\mathcal{L}\left(e^{-t/2}\sin\frac{t\sqrt{3}}{2}\right) \\ &= \mathcal{L}\left(e^{-t/2}\cos\frac{t\sqrt{3}}{2} - \frac{1}{\sqrt{3}}e^{-t/2}\sin\frac{t\sqrt{3}}{2}\right),\end{aligned}$$

$$\frac{2}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+3/4} = \mathcal{L}\left(\frac{2}{\sqrt{3}}e^{-t/2}\sin\frac{t\sqrt{3}}{2}\right),$$

and

$$\frac{2}{\sqrt{3}}e^{-s}\frac{\sqrt{3}/2}{(s+1/2)^2+3/4} = \mathcal{L}\left(\frac{2}{\sqrt{3}}e^{-(t-1)/2}\sin\frac{(t-1)\sqrt{3}}{2}H(t-1)\right),$$

Hence

$$\begin{aligned}y(t) = e^{-t/2}&\left(\cos\frac{t\sqrt{3}}{2} - \frac{1}{\sqrt{3}}\sin\frac{t\sqrt{3}}{2} + \frac{2}{\sqrt{3}}\sin\frac{t\sqrt{3}}{2}\right) \\ &- \frac{2}{\sqrt{3}}e^{-(t-1)/2}\sin\left(\frac{(t-1)\sqrt{3}}{2}\right)H(t-1)\end{aligned}$$

by uniqueness and linearity.

Answer: $y(t) = e^{-t/2}\left(\cos\frac{t\sqrt{3}}{2} + \frac{1}{\sqrt{3}}\sin\frac{t\sqrt{3}}{2}\right) - \frac{2}{\sqrt{3}}e^{-(t-1)/2}\sin\left(\frac{(t-1)\sqrt{3}}{2}\right)H(t-1)$, $t \geq 0$.

6. There are at least two different possibilites for solving this problem.

Alternative 1: consider that

$$\frac{\sin^2 2\omega}{1+\omega^2} = \frac{1-\cos 4\omega}{2(1+\omega^2)} = \frac{1}{2}\cdot\frac{1-\cos 4\omega}{i\omega}\cdot\frac{i\omega}{1+\omega^2} = \frac{1}{2}\cdot F(\omega)\cdot\overline{G(\omega)},$$

where

$$F(\omega) = \frac{1-\cos 4\omega}{i\omega} = \mathcal{F}\left(\underbrace{\frac{1}{2}\operatorname{sgn}(x)\left(H(x+4)-H(x-4)\right)}_{=f(x)}\right)$$

and

$$G(\omega) = \frac{-i\omega}{1+\omega^2} = \mathcal{F}\left(\underbrace{\frac{1}{2}\operatorname{sgn}(x)e^{-|x|}}_{=g(x)}\right).$$

We note that f and g both belong to $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, so by Plancherel's identity

$$\int_{-\infty}^{\infty} F(\omega)\overline{G(\omega)}\,d\omega = 2\pi\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \frac{\pi}{4}\int_{-4}^{4}\operatorname{sgn}(x)^2 e^{-|x|}\,dx = \frac{\pi}{2}\int_0^4 e^{-x}\,dx$$
$$= \frac{\pi}{2}\left(1-e^{-4}\right).$$

Alternative 2: using the same trigonometric identity as in the previous alternative, we find that

$$\int_{-\infty}^{\infty}\frac{\sin^2 2\omega}{1+\omega^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{1+\omega^2} - \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos 4\omega}{1+\omega^2}.$$

The first integral in the right-hand side is well-known:

$$\int_{-\infty}^{\infty}\frac{d\omega}{1+\omega^2} = [\arctan\omega]_{-\infty}^{\infty} = \pi.$$

In the second integral, we let $\xi = 4\omega$ so that

$$\int_{-\infty}^{\infty}\frac{\cos 4\omega}{1+\omega^2}\,d\omega = \frac{1}{4}\int_{-\infty}^{\infty}\frac{\cos\xi}{1+(\xi/4)^2}\,d\xi = \frac{1}{4}\int_{-\infty}^{\infty}\frac{16\cos\xi}{16+\xi^2}\,d\xi = 4\int_{-\infty}^{\infty}\frac{\cos\xi}{16+\xi^2}\,d\xi$$
$$= 4\int_{-\infty}^{\infty}\frac{\cos\xi}{16+\xi^2}\,d\xi = 4\int_{-\infty}^{\infty}\frac{\cos\xi}{16+\xi^2}\,d\xi$$
$$= 4\operatorname{Re}\left(\int_{-\infty}^{\infty}\frac{e^{-i\xi\cdot 1}}{16+\xi^2}\,d\xi\right) = 4\operatorname{Re}\left(\mathcal{F}\left(\frac{1}{4^2+\xi^2}\right)(x=1)\right)$$
$$= 4\operatorname{Re}\left(\frac{\pi}{4}e^{-4\cdot|1|}\right) = \pi e^{-4},$$

where we used the fact that $\dfrac{\cos\xi}{16+\xi^2}$ is real valued and that $\operatorname{Re}(e^{-i\xi}) = \cos\xi$. Hence

$$\int_{-\infty}^{\infty}\frac{\sin^2 2\omega}{1+\omega^2} = \frac{1}{2}\left(1-e^{-4}\right).$$

Answer: $\frac{1}{2}\left(1-e^{-4}\right)$.

7. Note that $e^{-x^2/2}$ and $xe^{-x^2/2}$ belong to $G(\mathbf{R})$, so if $u(x) = e^{-x^2/2}$ and $U(\omega) = \mathcal{F}\,u(\omega)$, then

$$U'(\omega) = -i\,\mathcal{F}(xe^{-x^2/2})(\omega) = -i\int_{-\infty}^{\infty} xe^{-x^2/2}e^{-i\omega x}\,dx$$

$$= /\ \text{I.B.P.}\ / = -i\left[-e^{-x^2}e^{-i\omega x}\right]_{-\infty}^{\infty} - i\int_{-\infty}^{\infty}\frac{-i\omega}{2}\,e^{-x^2/2}e^{-i\omega x}\,dx = -\omega U(\omega).$$

So U must satisfy

$$U'(\omega) + \omega U(\omega) = 0 \quad\Leftrightarrow\quad \frac{d}{d\omega}\left(e^{\omega^2/2}U(\omega)\right) = 0 \quad\Leftrightarrow\quad U(\omega) = Ce^{-\omega^2/2}.$$

However, we can only have one Fourier transform so we need to find a value for C. It is clear that

$$U(0) = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = /\ x = t\sqrt{2}\ / = \sqrt{2}\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{2}\sqrt{\pi} = \sqrt{2\pi}.$$

Therefore $C = U(0) = \sqrt{2\pi}$ and we have shown that

$$\mathcal{F}\left(e^{-x^2/2}\right)(\omega) = \sqrt{2\pi}e^{-\omega^2/2}.$$

Answer: see above.