Transform theory 2022-10-21 – Solutions

1. (a) Yes. This follows since $\overline{u(x)} = u(x)$ and $\overline{e^{i\omega x}} = e^{-i\omega x}$. See the Lecture notes.
   (b) Yes. The function is continuous and decays rapidly as $|x| \to \infty$.
   (c) No. This is not true in general. We need more information about the series of the derivatives u_k'. A famous counter example is Weierstrass' function (continuous but nowhere differentiable).
   (d) Yes. The Fourier series *is* $u(x) \sim \sin 2x$ for $x \in \mathbf{R}$. Obviously the convergence of a series with one term is uniform.
   (e) No. The function does not tend to zero as $|s| \to \infty$ (it's not even bounded as $|s| \to \infty$).

   Answer: Yes, Yes, No, Yes, No.

2. We're looking for exponentially bounded solutions, so we assume that $|u(t)| \leq Ke^{at}$ for some constants $a, C > 0$. Taking the Laplace transform, we obtain that

$$s^2U(s) - su(0) - u'(0) + 3\big(sU(s) - u(0)\big) + 2U(s) = \frac{3}{s+1}$$
$$\Leftrightarrow \quad U(s)\left(s^2 + 3s + 2\right) = 3s + 7 + \frac{3}{s+1}, \quad \operatorname{Re} s > -1.$$

   Hence,

$$U(s) = \frac{(3s+7)(s+1)+3}{(s^2+3s+2)(s+1)} = \frac{3s^2+10s+10}{(s+1)^2(s+2)}.$$

   Decomposition into partial fractions yields

$$U(s) = \frac{1}{s+1} + \frac{3}{(s+1)^2} + \frac{2}{s+2}.$$

   From a table,

$$\mathcal{L}(e^{-t}) = \frac{1}{s+1}, \quad \mathcal{L}(te^{-t}) = \frac{1}{(s+1)^2}, \quad \mathcal{L}(e^{-2t}) = \frac{1}{s+2},$$

   so

$$u(t) = e^{-t} + 3te^{-t} + 2e^{-2t}$$

   by uniqueness and linearity. Obviously u is exponentially bounded.

   Answer: $u(t) = (1+3t)e^{-t} + 2e^{-2t}, \quad t \geq 0.$

3. The sum in the equation is the convolution of u with the function $k \mapsto 2^k$. We take the Z transform of the equation and find that

$$z^2U(z) - z^2u[0] - zu[1] + U(z)\frac{z}{z-2} = \frac{z^2}{z^2+1},$$

   where we assume that (at least) $|z| > 2$. Reformulating this equation, we find that

$$U(z)\left(z^2 + \frac{z}{z-2}\right) = \frac{z^2}{z^2+1} \quad \Leftrightarrow \quad U(z) = \frac{z^2(z-2)}{(z^2+1)(z^2(z-2)+z} = \frac{(z-2)z}{(z^2+1)(z-1)^2}.$$

We decompose into partial fractions:

$$\frac{U(z)}{z} = \frac{-1/2}{(z-1)^2} + \frac{1}{z-1} - \frac{1}{z^2+1} - \frac{z/2}{z^2+1}.$$

From a table:

$$\mathcal{Z}\,(1) = \frac{z}{z-1}. \quad \mathcal{Z}\,(k) = \frac{z}{(z-1)^2}, \quad \mathcal{Z}\left(\cos\frac{n\pi}{2}\right) = \frac{z^2}{z^2+1}, \quad \mathcal{Z}\left(\sin\frac{n\pi}{2}\right) = \frac{z}{z^2+1}.$$

By linearity and uniqueness, we therefore find that

$$u[k] = 1 - \frac{1}{2}k - \cos\frac{n\pi}{2} - \frac{1}{2}\sin\frac{n\pi}{2}.$$

Answer: $u[k] = 1 - \frac{1}{2}k - \cos\frac{n\pi}{2} - \frac{1}{2}\sin\frac{n\pi}{2}$, $k = 0, 1, 2, 3, \ldots$.

4. Clearly $u \in E$. We find that, for $k \neq 0$,

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \pi e^{-ikx}\,dx = \frac{1}{2}\left[\frac{e^{-ikx}}{-ik}\right]_{-\pi/2}^{\pi/2}$$
$$= \frac{1}{2ik}\left(-e^{-ik\pi/2} + e^{ik\pi/2}\right) = \frac{1}{k}\sin\left(\frac{k\pi}{2}\right)$$

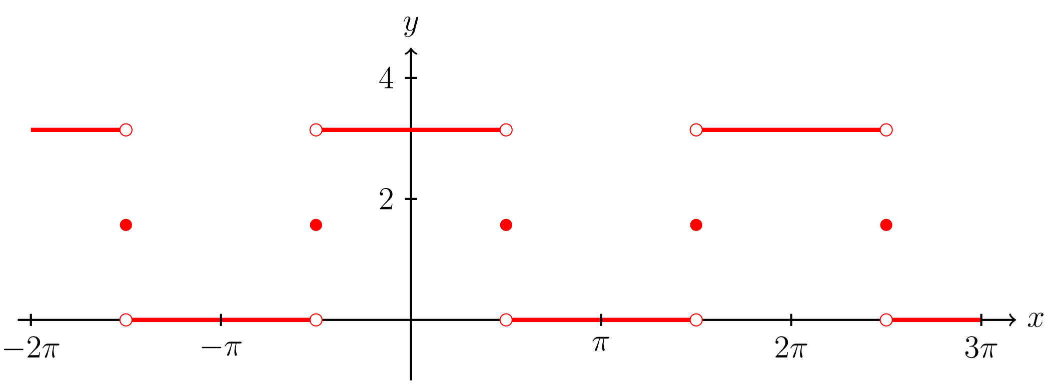
and

$$c_0 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \pi\,dx = \frac{\pi}{2}.$$

Hence

$$u(x) \sim \frac{\pi}{2} + \sum_{k\neq 0}\frac{1}{k}\sin\left(\frac{k\pi}{2}\right)e^{ikx} = \frac{\pi}{2} + \sum_{m=-\infty}^{\infty}\frac{(-1)^m}{2m+1}e^{i(2m+1)x}$$
$$= \frac{\pi}{2} + 2\sum_{m=0}\frac{(-1)^m}{2m+1}\cos((2m+1)x).$$

Furthermore, the function u is differentiable for $x \neq (2m+1)\pi/2$, $m \in \mathbf{Z}$, and the function has right- and lefthand derivatives at $x = (2m+1)\pi$ (it is piecewise constant). Hence – by Dirichlet's theorem – the Fourier series is convergent and converges to $u(x)$ for $x \neq (2m+1)\pi/2$ and to $\pi/2$ at $x = (2m+1)\pi$ ($m \in \mathbf{Z}$).



In particular,

$$\begin{aligned}\pi = u(0) &= \frac{\pi}{2} + \sum_{k\neq 0} \frac{1}{k} \sin\left(\frac{k\pi}{2}\right) = \frac{\pi}{2} + 2\sum_{m=0} \frac{1}{2m+1} \sin\left(\frac{(2m+1)\pi}{2}\right) \\ &= \frac{\pi}{2} + 2\sum_{m=0} \frac{(-1)^m}{2m+1}\end{aligned}$$

since $\frac{1}{k} \sin\left(\frac{k\pi}{2}\right)$ is even with respect to k, $\sin(2m\pi/2) = 0$ and $\sin((2m+1)\pi/2) = (-1)^m$ for $m \in \mathbf{Z}$. Hence

$$\sum_{m=0} \frac{(-1)^m}{2m+1} = \frac{\pi}{4}.$$

Answer: $u(x) \sim \frac{\pi}{2} + \sum_{k\neq 0} \frac{1}{k} \sin\left(\frac{k\pi}{2}\right) e^{ikx}$; see above.

5. We're looking for a solution to $y''(x) = -y'(x) - 2y(x-\pi) + 2\sin^2 x$, so obviously y must be (at least) two times differentiable. Hence y' is continuous. This means that y'' must be continuous (since y solves the equation). Hence $y \in C^2$. Which means that $y'' \in C^2$, so $y \in C^4$ and so on. In other words, the solution must be very smooth.

   - $y \in C^3$ implies that the Fourier series of y, y' and y'' converges to $y(x)$, $y'(x)$ and $y''(x)$, respectively (Dirichlet's theorem). So, let $y(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$.
   - y being 2π-periodic and $y' \in E$ means we can form the termwise derivative of y (with equality due to the first point):

   $$y'(x) = \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}.$$

   - Similarly, $y'' \in E$, so (with equality since $y \in C^3$)

   $$y''(x) = \sum_{k=-\infty}^{\infty} -k^2 c_k e^{ikx}.$$

   Therefore, we can write

   $$\begin{aligned}y''(x) + y'(x) + 2y(x-\pi) = 2\sin^2 x \;&\Leftrightarrow\; \sum_{k=-\infty}^{\infty} (-k^2 + ik + 2e^{-ik\pi})c_k e^{ikx} = 2\left(\frac{1-\cos 2x}{2}\right) \\ &\Leftrightarrow\; \sum_{k=-\infty}^{\infty} (-k^2 + ik + 2(-1)^k)c_k e^{ikx} = 1 - \frac{1}{2}\left(e^{-i2x} + e^{i2x}\right).\end{aligned}$$

   For y to be a solution to the differential equation, we must therefore (by uniqueness) have:

   $$k^2 = ik + 2(-1)^k \quad \text{or} \quad c_k = 0, \quad \text{for } k \neq 0, \pm 2.$$

Obviously $k^2 \neq ik + 2(-1)^k$ if $k \neq 0$ since the left-hand side if real. Hence $c_k = 0$ for $k = \pm 1, \pm 3, \pm 4, \ldots$. For $k = 0$, we must have

$$(-0^2 + i \cdot 0 + 2)c_0 = 1 \quad \Leftrightarrow \quad c_0 = \frac{1}{2}.$$

If $k = \pm 2$, then

$$\begin{aligned}\big(-(\pm 2)^2 \pm 2i + 2(-1)^{\pm 2}\big)c_{\pm 2} = \frac{1}{2} \quad &\Leftrightarrow \quad \big(2 \pm 2i\big)c_{\pm 2} = \frac{1}{2} \\ &\Leftrightarrow \quad c_{\pm 2} = \frac{1}{4 \pm 4i} = \frac{4 \mp 4i}{32} = \frac{1 \mp i}{8}.\end{aligned}$$

Thus

$$y(x) = c_0 + c_{-2}e^{-i2x} + c_2 e^{i2x} = \frac{1}{2} + \frac{1}{8}\left((1-i)e^{-i2x} + (1+i)e^{i2x}\right) = \frac{1}{2} + \frac{1}{4}\left(\cos 2x - \sin 2x\right).$$

Answer: $y(x) = \frac{1}{2} + \frac{1}{4}(\cos 2x - \sin 2x)$.

6. (a) We observe that $f \in G(\mathbf{R})$ so the Fourier transform exists and if $\omega \neq \pm 1$, then

$$\begin{aligned}F(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \int_{-\pi}^{\pi} \cos(x)e^{-i\omega x}\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \left(e^{ix} + e^{-ix}\right)e^{-i\omega x}\,dx \\ &= \frac{1}{2}\left[\frac{e^{ix(1-\omega)}}{i(1-\omega)} + \frac{e^{-ix(1+\omega)}}{-i(1+\omega)}\right]_{x=-\pi}^{x=\pi} \\ &= \frac{1}{2}\left(\frac{e^{i\pi(1-\omega)}}{i(1-\omega)} + \frac{e^{-i\pi(1+\omega)}}{-i(1+\omega)} - \frac{e^{-i\pi(1-\omega)}}{i(1-\omega)} - \frac{e^{i\pi(1+\omega)}}{-i(1+\omega)}\right) \\ &= -\frac{1}{2}\left(\frac{e^{-i\pi\omega}}{i(1-\omega)} + \frac{e^{-i\pi\omega}}{-i(1+\omega)} - \frac{e^{i\pi\omega}}{i(1-\omega)} - \frac{e^{i\pi\omega}}{-i(1+\omega)}\right) \\ &= -\frac{1}{2}\left(\frac{-2\sin(\pi\omega)}{1-\omega} + \frac{2\sin(\pi\omega)}{1+\omega}\right) = \sin(\pi\omega)\left(\frac{1}{1-\omega} - \frac{1}{1+\omega}\right) = \frac{2\omega\sin(\pi\omega)}{1-\omega^2}.\end{aligned}$$

At $\omega = \pm 1$, we can either use continuity of F to define $F(\pm 1)$ or calculate directly:

$$F(\pm 1) = \int_{-\infty}^{\infty} u(x)e^{-ix}\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\left(1 + e^{\mp i2x}\right)\,dx = \pi.$$

(b) First, we note that

$$\begin{aligned}\int_{-t}^{t} \frac{\omega\sin(\pi\omega)}{1-\omega^2}\cos(\omega\xi)\,d\omega &= \int_{-t}^{t} \frac{\omega\sin(\pi\omega)}{1-\omega^2}\left(\cos(\omega\xi) + i\sin(\omega\xi)\right)\,d\omega \\ &= \int_{-t}^{t} \frac{\omega\sin(\pi\omega)}{1-\omega^2}\,e^{i\omega\xi}\,d\omega\end{aligned}$$

since the integrand is an even function so we can multiply by an odd funtion and have that part disappear when integrating symmetrically around the origin. We recognize that this limit is of the same type that is used for inverting the Fourier transform for the function u from the previous exercise. Moreover, for $x \neq \pm\pi$, the function u is differentiable whereas at $x = \pm\pi$, the function u has right- and lefthand derivatives. Hence by Dirichlet's theorem, we find that

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} U(\omega) e^{i\omega x}\, d\omega = \frac{u(x^+) + u(x^-)}{2} = \begin{cases} \cos x, & |x| < \pi, \\ (0-1)/2 = -1/2, & x = \pm\pi, \\ 0, & |x| > \pi. \end{cases}$$

Using this result, we find that

$$\lim_{t\to\infty} \int_{-t}^{t} \frac{\omega \sin(\pi\omega)}{1-\omega^2} e^{i\omega\xi}\, d\omega = \begin{cases} \pi\cos\xi, & |\xi| < \pi, \\ \pi(0-1)/2 = -\pi/2, & \xi = \pm\pi, \\ 0, & |\xi| > \pi. \end{cases}$$

Answer: (a) $U(\omega) = \begin{cases} \frac{2\omega\sin(\pi\omega)}{1-\omega^2}, & \omega \neq \pm 1, \\ \pi, & \omega = \pm 1; \end{cases}$ (b) $\begin{cases} \pi\cos\xi, & |\xi| < \pi, \\ -\pi/2, & \xi = \pm\pi, \\ 0, & |\xi| > \pi. \end{cases}$

7. Using the formula for the Laplace transform of a periodic function, we find that

$$U(s) = \frac{1}{1-e^{-s}} \int_0^1 t^2 e^{-st}\, dt = \frac{e^s}{e^s - 1} \int_0^1 t^2 e^{-st}\, dt.$$

Letting $s = 2 + 2\pi ik$, we find that

$$U(2+2\pi ik) = \frac{e^{2+2\pi ik}}{e^{2+2\pi ik} - 1} \int_0^1 t^2 e^{-(2+2\pi ik)t}\, dt = \frac{e^2}{e^2 - 1} \int_0^1 t^2 e^{-2t} e^{-2\pi ikt}\, dt.$$

Here we recognize that the integral is the Fourier coefficient c_k, $k \in \mathbf{Z}$, for the function $f(t) = t^2 e^{-2t}$, $0 \le t < 1$ (periodically extended with period 1). Thus we have shown that

$$U(2+2\pi ik) = \frac{e^2 c_k}{e^2 - 1}$$

and since $f \in E$ has one-sided derivatives at all points, Dirichlet's theorem proves that

$$\sum_{k=-N}^{N} \frac{e^2 c_k}{e^2-1} = \frac{e^2}{e^2-1} \sum_{k=-N}^{N} c_k = \frac{e^2}{e^2-1} \sum_{k=-N}^{N} c_k e^{ik\cdot 0}$$
$$\to \frac{e^2}{e^2-1} \cdot \frac{f(0^-) + f(0^+)}{2} = \frac{e^2}{e^2-1} \cdot \frac{0 + 1^2 e^{-2}}{2} = \frac{1}{2(e^2-1)}, \quad \text{as } N \to \infty.$$

Answer: $\dfrac{1}{2(e^2-1)}$.