Transform theory 2024-06-01 – Solutions

1. (a) No. Since $u_n(x) \to u(x)$, where $u(x) = 0$ for $|x| < 1$ and $u(\pm 1) = 1$, so u is discontinuous, the convergence can't be uniform since each u_n is continuous.

   (b) No, $U(\omega)$ is discontinuous at $\omega = 0$.

   (c) Yes. Fourier series of functions in E always converge in $L^2(-\pi, \pi)$.

   (d) Yes, $U(z) = \dfrac{1}{z^2}\dfrac{z}{z+1} = \mathcal{Z}\left((-1)^{k-2}H[k-2]\right)$.

   (e) Yes, $F(s) = \mathcal{L}\left(\dfrac{1}{2}\sinh(2(t-1))H(t-1)\right)$, $\operatorname{Re} s > 2$.

   Answer: No. No. Yes. Yes. Yes.

2. (a) We find that

$$\mathcal{L}\,u(s) = \int_0^\infty e^{-3t}e^{-st}\,dt = \int_0^\infty e^{-(s+3)t}\,dt = \left[\frac{e^{-(s+3)t}}{-(s+3)}\right]_0^\infty = \frac{1}{s+3}, \quad \operatorname{Re}(s+3) > 0.$$

   Note that $\operatorname{Re}(s+3) > 0 \Leftrightarrow \operatorname{Re} s > -3$.

   (b) We assume that y, y', y'' all belong to X_a (and verify this at the end). Taking the Laplace transform, we obtain that

$$s^2Y(s) - sy(0) - y'(0) - 2\big(sY(s) - y(0)\big) + 2Y(s) = \frac{16}{s+3}$$
$$\Leftrightarrow \quad Y(s)\left(s^2 + 2s - 3\right) = \frac{16}{s+3} + s - 1, \quad \operatorname{Re} s > -3.$$

   Hence, since $s^2 + 2s - 3 = (s+3)(s-1)$,

$$Y(s) = \frac{16}{(s+3)^2(s-1)} + \frac{s-1}{(s+3)(s-1)} = \left(\frac{1}{s-1} - \frac{4}{(s+3)^2} - \frac{1}{s+3}\right) + \frac{1}{s+3}$$
$$= \frac{1}{s-1} - \frac{4}{(s+3)^2}.$$

   From a table,

$$\mathcal{L}(e^t) = \frac{1}{s-1} \quad \text{and} \quad \mathcal{L}(te^{-3t}) = \frac{1}{(s+3)^2},$$

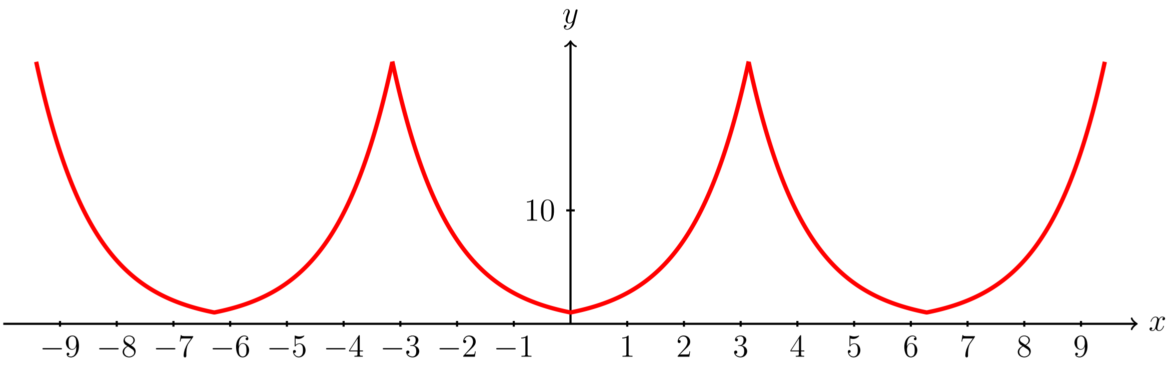
   so

$$y(t) = e^t - 4te^{-3t}$$

   by uniqueness and linearity. Obviously y and its derivatives are exponentially bounded.

   Answer: $y(t) = e^t - 4te^{-3t}, \quad t > 0$.

3. Clearly $u \in E$. This is obvious since the the periodic extension function is continuous everywhere ($e^{|-\pi|} = e^{\pi}$ at the end points). There are no jumps. Furthermore, u is infinitely differentiable for $x \neq n\pi$, and at $x = n\pi$ the right- and lefthand derivatives exist. Hence – by Dirichlet's theorem – the Fourier series of u is convergent and converges to $u(x)$ for all $x \in \mathbf{R}$. The convergence is uniform since $u(-\pi) = u(\pi)$ and $u' \in E$. We sketch the graph of the Fourier series below.



We need the Fourier coefficients, so we observe that u is an even function so a pure cosine-series is sufficient. The integrand is even so for $k \geq 0$,

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{|x|}\cos kx\,dx = \frac{2}{\pi}\int_0^{\pi} e^x \cos kx\,dx = \frac{1}{\pi}\int_0^{\pi} e^x\left(e^{ikx}+e^{-ikx}\right)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi}\left(e^{x(ik+1)}+e^{x(1-ik)}\right)\,dx = \frac{1}{\pi}\left[\frac{e^{x(1+ik)}}{1+ik}+\frac{e^{x(1-ik)}}{1-ik}\right]_0^{\pi} \\ &= \frac{1}{\pi}\left(\frac{e^{ik\pi}e^{\pi}}{1+ik}+\frac{e^{-ik\pi}e^{\pi}}{1-ik}-\frac{1}{1+ik}-\frac{1}{1-ik}\right) \\ &= \frac{1}{\pi}\left(\frac{(-1)^k e^{\pi}-1}{1+ik}+\frac{(-1)^k e^{\pi}-1}{1-ik}\right) = \frac{2(-1)^k e^{\pi}-2}{\pi(1+k^2)}. \end{aligned}$$

Thus $u(x) = \dfrac{e^{\pi}-1}{\pi} + \displaystyle\sum_{k=1}^{\infty}\frac{2(-1)^k e^{\pi}-2}{\pi(1+k^2)}\cos kx$ with equality due to the argument above.

Answer: $u(x) = \dfrac{e^{\pi}-1}{\pi} + \dfrac{2}{\pi}\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k e^{\pi}-1}{1+k^2}\cos kx$; see above.

4. The left-hand side is a convolution of y with $t \mapsto e^{-2|t|}$ (which is a bounded function in $G(\mathbf{R})$), so taking the Fourier transform (assuming that $y \in G(\mathbf{R})$) shows that

$$Y(\omega)\frac{2\cdot 2}{2^2+\omega^2} = \sqrt{\pi}e^{-\omega^2/4} \quad\Leftrightarrow\quad Y(\omega) = \frac{1}{4}\left(4+\omega^2\right)\sqrt{\pi}e^{-\omega^2/4}.$$

Note that

$$\omega^2\sqrt{\pi}e^{-\omega^2/4} = -(i\omega)^2\sqrt{\pi}e^{-\omega^2/4}$$

so

$$\begin{aligned} y(t) &= e^{-x^2} - \frac{1}{4}\left(\frac{d}{dt}\right)^2 e^{-x^2} = e^{-x^2} - \frac{1}{4}\frac{d}{dt}\left(-2xe^{-x^2}\right) = e^{-x^2} - \frac{1}{4}\left(-2e^{-x^2}+4x^2e^{-x^2}\right) \\ &= \left(\frac{3}{2}-x^2\right)e^{-x^2} \end{aligned}$$

by linearity and uniqueness (obviously y is differentiable and absolutely integrable).

Answer: $y(t) = \left(\dfrac{3}{2}-x^2\right)e^{-x^2}$, $t \in \mathbf{R}$.

5. Taking the Z transform with $|z| > 2$ yields

$$\begin{aligned} z^2U(z) - (z^2u[0] + zu[1]) - (zU(z) - zu[0]) - 2U(z) &= \frac{6z}{(z-2)^2} - \frac{z}{z-2} + 1 \\ \Leftrightarrow \quad \left(z^2 - z - 2\right) U(z) &= 3z^2 + \frac{6z - z^2 + 2z + z^2 - 4z + 4}{(z-2)^2}. \end{aligned}$$

We note that

$$\frac{6z - z^2 + 2z + z^2 - 4z + 4}{(z-2)^2} = \frac{4(z+1)}{(z-2)^2}$$

and since $z^2 - z - 2 = (z-2)(z+1)$, so

$$U(z) = \frac{3z^2}{(z-2)(z+1)} + \frac{4}{(z-2)^3}, \quad |z| > 2.$$

We decompose the first term into partial fractions,

$$z \cdot \frac{3z}{(z-2)(z+1)} = z\left(\frac{1}{z+1} + \frac{2}{z-2}\right) = \frac{z}{z+1} + 2 \cdot \frac{z}{z-2}$$

and rewrite the second term as

$$\frac{4}{(z-2)^3} = \frac{1}{z} \cdot \frac{4z}{(z-2)^3}$$

so we can use a table (and uniqueness) to find that

$$\begin{aligned} u[n] &= (-1)^n + 2 \cdot 2^n + \binom{n-1}{2} 2^{n-1} H[n-1] \\ &= (-1)^n + 2^{n+1} + \frac{(n-1)(n-2)}{2} 2^{n-1} H[n-1] \\ &= (-1)^n + 2^{n+1} + (n^2 - 3n + 2)2^{n-2} H[n-1]. \end{aligned}$$

We see that $u[0] = u[1] = 3$ and we can directly verify that this solves the equation.

Answer: $u[n] = (-1)^n + 2^{n+1} + (n^2 - 3n + 2)2^{n-2} H[n-1]$, $n = 0, 1, 2, \ldots$

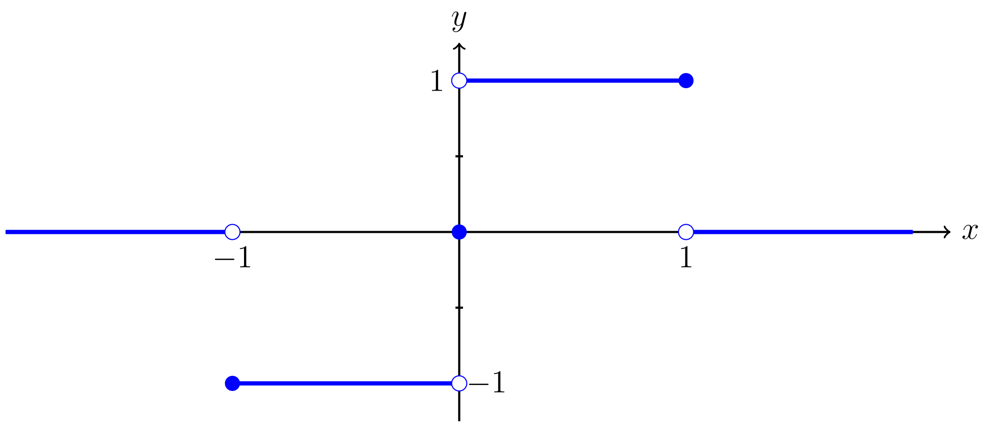
6. (a) We observe that $f \in G(\mathbf{R})$ so the Fourier transform exists and

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx = \int_{-1}^{1} \operatorname{sgn}(x) e^{-i\omega x}\, dx \\ &= \int_{-1}^{0} -1 \cdot e^{-i\omega x}\, dx + \int_{0}^{1} 1 \cdot e^{-i\omega x}\, dx \\ &= \left[-\frac{e^{-i\omega x}}{-i\omega}\right]_{-1}^{0} + \left[\frac{e^{-i\omega x}}{-i\omega}\right]_{0}^{1} = \frac{1 - e^{i\omega}}{i\omega} + \frac{1 - e^{-i\omega}}{i\omega} = \frac{2 - (e^{i\omega} + e^{-i\omega})}{i\omega} \\ &= \frac{2 - 2\cos\omega}{i\omega} = 2i \cdot \frac{\cos\omega - 1}{\omega}, \quad \omega \neq 0. \end{aligned}$$

At $\omega = 0$, we calculate directly:

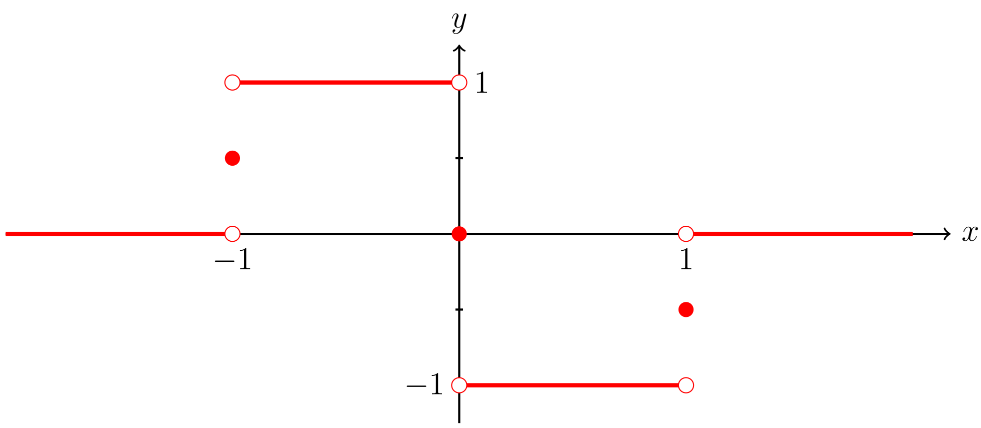
$$F(0) = \int_{-\infty}^{\infty} f(x) e^{-i \cdot 0 \cdot x}\, dx = \int_{-1}^{1} \operatorname{sgn}(x)\, dx = 0.$$

(b) Drawing the graph of $f(x)$, we find the following.



Since $D^{\pm}f(x)$ exists for every $x \in \mathbf{R}$, Fourier inversion (Dirichlet's theorem for the Fourier transform) yields

$$\begin{aligned}
\lim_{R\to\infty} \frac{1}{i\pi}\int_{-R}^{R} \frac{\cos\omega - 1}{\omega} e^{i\omega x}\, d\omega &= -\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R}\frac{2i(\cos\omega-1)}{\omega}e^{i\omega x}\,d\omega \\
&= -\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R} F(\omega)e^{i\omega x}\,d\omega \\
&= -\frac{u(x^+)+u(x^-)}{2} = \begin{cases} -\operatorname{sgn}(x), & |x|<1, \\ \pm\frac{1}{2}, & x = \mp 1, \\ 0, & |x|>1. \end{cases}
\end{aligned}$$



(c) Note that with $F(\omega)$ from (a), for $\omega \neq 0$,

$$|F(\omega)|^2 = \frac{4(\cos\omega - 1)^2}{\omega^2}$$

and since $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, we can use Plancherel's theorem:

$$\int_{-\infty}^{\infty} \frac{4(\cos\omega-1)^2}{\omega^2}\,d\omega = 2\pi\int_{-\infty}^{\infty} |f(x)|^2\,dx = 2\pi\int_{-1}^{1} 1\,dx = 4\pi,$$

so

$$\int_{-\infty}^{\infty}\frac{(\cos\omega-1)^2}{\omega^2}\,d\omega = \pi.$$

Answer:
(a) $F(\omega) = \dfrac{2i(\cos\omega - 1)}{\omega}$, $\omega \neq 0$, $F(0) = 0$
(b) $\operatorname{sgn}(x)$ when $|x| < 1$, $\pm 1/2$ when $x = \pm 1$, 0 when $|x| > 1$.
(c) se above.

7. Let $u_n(t) = \dfrac{n^2\cos(t/n)}{n^2 + 1 + n^2t^2}$ for $n = 1, 2, 3, \ldots$ We find the pointwise limit as $n \to \infty$:

$$u(t) = \lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} \frac{\cos(t/n)}{1 + 1/n^2 + t^2} = \frac{1}{1+t^2}.$$

Since

$$\begin{aligned}
|u_n(t) - u(t)| &= \left|\frac{n^2\cos(t/n)}{n^2+1+n^2t^2} - \frac{1}{1+t^2}\right| = \left|\frac{n^2\cos(t/n)(1+t^2) - (n^2+1+n^2t^2)}{(n^2+1+n^2t^2)(1+t^2)}\right| \\
&= \left|\frac{\cos(t/n)(1+t^2) - (1+t^2) - 1/n^2}{(1+1/n^2+t^2)(1+t^2)}\right| = \left|\frac{(1+t^2)(\cos(t/n)-1) - 1/n^2}{(1+1/n^2+t^2)(1+t^2)}\right| \\
&\leq \frac{\overbrace{(1+t^2)}^{\leq 2}\,\overbrace{|\cos(t/n)-1|}^{\leq |t/n| \leq 1/n} + 1/n^2}{\underbrace{(1+1/n^2+t^2)}_{\geq 1}\,\underbrace{(1+t^2)}_{\geq 1}} \leq \frac{2}{n} + \frac{1}{n^2}
\end{aligned}$$

it is clear that

$$\sup_{0\leq t\leq 1} |u_n(t) - u(t)| \leq \frac{2}{n} + \frac{1}{n^2} \to 0, \quad \text{as } n \to \infty$$

so the convergence is uniform. Thus

$$\lim_{n\to\infty} \int_0^1 u_n(t)\,dt = \int_0^1 u(t)\,dt = \int_0^1 \frac{1}{1+t^2}\,dt = \big[\arctan t\big]_0^1 = \frac{\pi}{4}.$$

Answer: $\dfrac{\pi}{4}$.