Transform theory 2024-08-22 – Solutions

1. (a) Yes. Since $u_n(x) \to 0$ for $|x| \le 1/2$ and $|u_n(x) - 0| = |x|^n \le 2^{-n} \to 0$ independent of $x \in [-1/2, 1/2]$.
   (b) No, $U(\omega)$ is discontinuous at $\omega = 0$.
   (c) No. The function does not tend to zero as $|s| \to \infty$ (it's not even bounded as $|s| \to \infty$).
   (d) Yes. If $\mathcal{Z}\,u = \mathcal{Z}\,v$ for $|z| > R > 0$ (some R), then $u[n] = v[n]$ for all $n = 0, 1, 2, \ldots$
   (e) Yes. Note that $\cos x \sin x = \frac{1}{2} \sin 2x$ and that $\mathcal{L}(\sin 2x) = \frac{2}{s^2+4}$, $\operatorname{Re} s > 0$.

   Answer: Yes. No. No. Yes. Yes.

2. (a) We find that

$$\mathcal{Z}(3^n)(z) = \sum_{n=0}^{\infty} 3^n\, z^{-n} = \sum_{n=0}^{\infty} (z/3)^{-n} = \frac{1}{1 - 3/z} = \frac{z}{z-3}, \quad |z| > 3.$$

   (b) Taking the Z transform with $|z| > 3$ yields

$$z^2 U(z) - (z^2 u[0] + z u[1]) - 3(zU(z) - zu[0]) + 2U(z) = \frac{4z}{z-3}$$

$$\Leftrightarrow \quad \left(z^2 - 3z + 2\right) U(z) = 2z^2 - z + \frac{4z}{z-3} = z \cdot \frac{2z^2 - 7z + 7}{z-3}.$$

   Thus, since $z^2 - 3z + 2 = (z-1)(z-2)$,

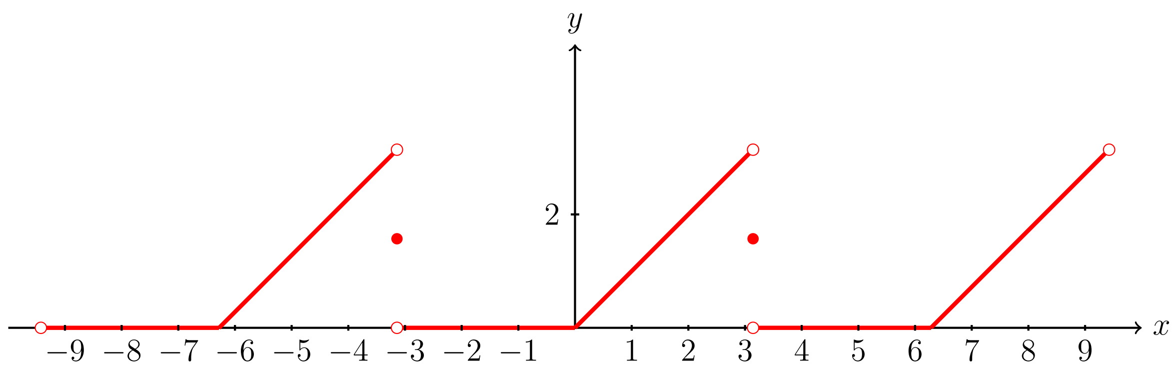
$$U(z) = z \cdot \frac{2z^2 - 7z + 7}{(z-1)(z-2)(z-3)} = z \left(\frac{2}{z-3} + \frac{-1}{z-2} + \frac{1}{z-1} \right),$$

   where we decomposed into partial fractions. We can now use a table (and uniqueness) to find that

$$u[n] = 2 \cdot 3^n - 2^n + 1.$$

   Answer: $u[n] = 2 \cdot 3^n - 2^n + 1$, $n = 0, 1, 2, \ldots$

3. Clearly $u \in E$. This is obvious since the the periodic extension function is continuous everywhere except for the jumps at odd multiples of π. Furthermore, u is infinitely differentiable for $x \ne n\pi$, and at $x = n\pi$ the right- and lefthand derivatives exist. Hence – by Dirichlet's theorem – the Fourier series of u is convergent and converges to $u(x)$ for all $x \ne (2m+1)\pi$. At $x = (2m+1)\pi$, the Fourier series converges to $\pi/2$. The convergence can't be uniform since the Fourier series converges to something that is discontinuous at $x = (2m+1)\pi$. We sketch the graph of the Fourier series below.



We find that, for $k \neq 0$,

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx = \frac{1}{2\pi}\int_0^{\pi} xe^{-ikx}\,dx = \frac{1}{2\pi}\left(\left[\frac{xe^{-ikx}}{-ik}\right]_0^{\pi} + \frac{1}{ik}\int_0^{\pi} e^{-ikx}\,dx\right)$$
$$= \frac{ie^{-ik\pi}}{2k} + \frac{1}{2\pi k^2}\left(e^{-ik\pi}-1\right) = i\frac{(-1)^k}{2k} + \frac{(-1)^k-1}{2\pi k^2}$$

and

$$c_0 = \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{\pi}{4}.$$

Hence

$$u(x) \sim \frac{\pi}{4} + \sum_{k\neq 0}\left(i\frac{(-1)^k}{2k} + \frac{(-1)^k-1}{2\pi k^2}\right)e^{ikx}.$$

Answer: $u(x) \sim \frac{\pi}{4} + \sum_{k\neq 0}\left(i\frac{(-1)^k}{2k} + \frac{(-1)^k-1}{2\pi k^2}\right)e^{ikx}$; see above.

4. The integral in the left-hand side is the one-sided convolution of u with $t \mapsto e^{3t} - e^t$, so taking the Laplace transform shows that

$$U(s) + U(s)\,\mathcal{L}(e^{3t}-e^t) = \frac{1}{s-2}, \quad \operatorname{Re} s > 2.$$

Thus

$$U(s)\left(1 + \frac{1}{s-3} - \frac{1}{s-1}\right) = \frac{1}{s-2} \quad \Leftrightarrow \quad U(s)\cdot\frac{s^2-4s+5}{(s-1)(s-3)} = \frac{1}{s-2}.$$

We solve for $U(s)$ and find that

$$U(s) = \frac{s^2-4s+3}{(s-2)(s^2-4s+5)} = \frac{-1}{s-2} + \frac{2s-4}{s^2-4s+5} = \frac{-1}{s-2} + 2\cdot\frac{s-2}{(s-2)^2+1}.$$

Since $\mathcal{L}(e^{2t}v(t)) = V(s-2)$, we find by uniqueness that

$$u(t) = -e^{2t} + 2e^{2t}\cos t, \quad t > 0.$$

Answer: $u(t) = -e^{2t} + 2e^{2t}\cos t$, $t > 0$.

5. We're looking for a solution to $y'(x) = 4y(x+\pi) + 1 - e^{i7x}$, so obviously y must be (at least) differentiable. Hence y is continuous. This means that y' must be continuous (since y solves the equation). Hence $y \in C^1$. Which means that $y' \in C^1$, so $y \in C^2$ and so on. In other words, the solution must be very smooth.

   - $y \in C^2$ implies that the Fourier series of y and y' converges to $y(x)$ and $y'(x)$, respectively (by Dirichlet's theorem). So, let $y(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$.
   - y being 2π-periodical and $y' \in E$ means we can form the termwise derivative of y (with equality due to the first point):

$$y'(x) = \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}.$$

Therefore, we can write

$$y'(x) - 4y(x+\pi) = 1 - e^{i7x} \quad \Leftrightarrow \quad \sum_{k=-\infty}^{\infty} (ik - 4e^{ik\pi})c_k e^{ikx} = 1 - e^{i7x}$$
$$\Leftrightarrow \quad \sum_{k=-\infty}^{\infty} (ik - 4(-1)^k)c_k e^{ikx} = 1 - e^{i7x}.$$

For y to be a solution to the differential equation, we must therefore (by uniqueness) have:

$$ik = 4(-1)^k \quad \text{or} \quad c_k = 0, \quad k \neq 0, 7.$$

Obviously $c_k = 0$ is the only possibility when $k \neq 0, 7$. If $k = 0$ we find that $-4c_0 = 1$, so $c_0 = -1/4$. If $k = 7$, then

$$(7i - 4(-1)^7)c_7 = -1 \quad \Leftrightarrow \quad c_7 = -\frac{1}{4+7i} = -\frac{4-7i}{65} = \frac{7i-4}{65}.$$

Hence our solutions must have the form

$$y(x) = c_0 + c_7 e^{i7x} = -\frac{1}{4} + \frac{7i-4}{65}\, e^{i7x}.$$

Answer: $y(x) = -\dfrac{1}{4} + \dfrac{7i-4}{65}\, e^{i7x}.$

6. (a) We observe that $f \in G(\mathbf{R})$ so the Fourier transform exists and

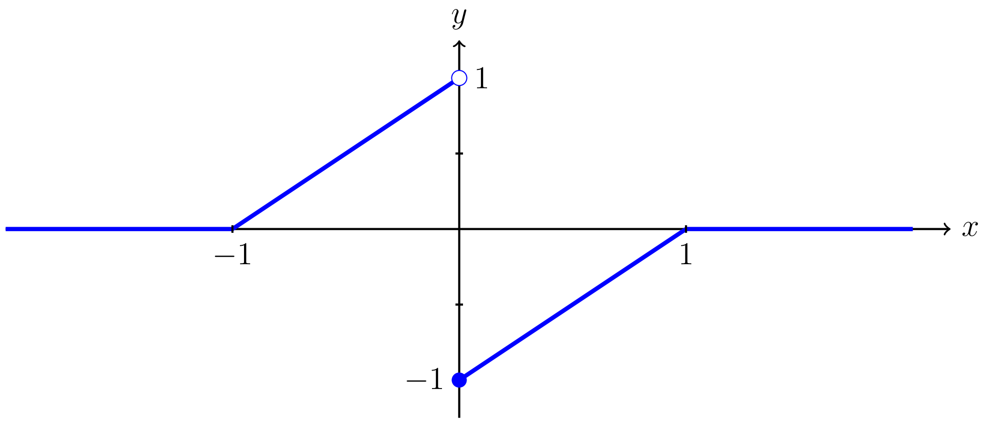
$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \\
&= \int_{-1}^{0} (1+x)\cdot e^{-i\omega x}\,dx + \int_{0}^{1} (x-1)\cdot e^{-i\omega x}\,dx \\
&= \left[\frac{(1+x)e^{-i\omega x}}{-i\omega}\right]_{-1}^{0} + \frac{1}{i\omega}\int_{-1}^{0} e^{-i\omega x}\,dx + \left[\frac{(x-1)e^{-i\omega x}}{-i\omega}\right]_{0}^{1} + \frac{1}{i\omega}\int_{0}^{1} e^{-i\omega x}\,dx \\
&= -\frac{2}{i\omega} + \frac{1}{i\omega}\int_{-1}^{1} e^{-i\omega x}\,dx = \frac{2i}{\omega} + \frac{1}{i\omega}\left[\frac{e^{-i\omega x}}{-i\omega}\right]_{-1}^{1} \\
&= \frac{2i\omega - (e^{i\omega} - e^{-i\omega})}{\omega^2} = 2i\frac{\omega - \sin\omega}{\omega^2}, \quad \omega \neq 0.
\end{aligned}$$

At $\omega = 0$, we calculate directly:

$$F(0) = \int_{-\infty}^{\infty} f(x)e^{-i\cdot 0\cdot x}\,dx = \int_{-1}^{1} f(x)\,dx = 0$$

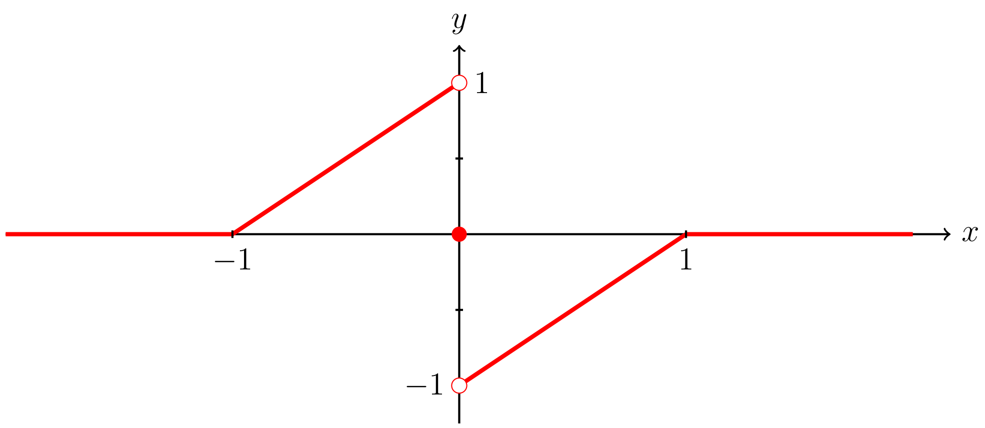
since the integrand is odd.

(b) Drawing the graph of $f(x)$, we find the following.



Since $D^{\pm}f(x)$ exists for every $x \in \mathbf{R}$, Fourier inversion (Dirichlet's theorem for the Fourier transform) yields

$$\begin{aligned}\lim_{R\to\infty} \frac{1}{i\pi} \int_{-R}^{R} \frac{\sin\omega - \omega}{\omega^2} e^{i\omega x}\, d\omega &= -\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \frac{2i(\sin\omega - \omega)}{\omega^2} e^{i\omega x}\, d\omega \\ &= \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} F(\omega) e^{i\omega x}\, d\omega \\ &= \frac{f(x^+) + f(x^-)}{2} = \begin{cases} 1+x, & -1 \le x < 0, \\ 0, & x = 0, \\ x-1, & 0 < x \le 1, \\ 0, & |x| > 1. \end{cases}\end{aligned}$$



(c) Note that with $F(\omega)$ from (a), for $\omega \neq 0$,

$$|F(\omega)|^2 = \frac{4(\omega - \sin\omega)^2}{\omega^4}$$

and since $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, we can use Plancherel's theorem:

$$\int_{-\infty}^{\infty} \frac{4(\omega - \sin\omega)^2}{\omega^4}\, d\omega = 2\pi \int_{-\infty}^{\infty} |f(x)|^2\, dx = 4\pi \int_{-1}^{0} (1+x)^2\, dx = \left[\frac{(1+x)^3}{3}\right]_{-1}^{0} = \frac{4\pi}{3},$$

so

$$\int_{-\infty}^{\infty} \frac{(t - \sin t)^2}{t^4}\, d\omega = \frac{\pi}{3}.$$

Answer:
(a) $F(\omega) = \dfrac{2i(\omega - \sin\omega)}{\omega^2}$, $\omega \neq 0$, $F(0) = 0$
(b) $f(x)$ when $|x| < 1$, 0 when $x = 0$ or $|x| \geq 1$.
(c) se above.

7. Let $u_k(x) = \dfrac{\cos kx}{x^2 + k^3}$. Clearly

$$|u_k(x)| \leq \frac{1}{k^3}, \quad k = 1, 2, 3, \ldots$$

so the series defining $u(x)$ is convergent for all x (actually uniformly convergent by the M-test). To show that $u(x)$ is differentiable, we prove that

$$\sum_{k=1}^{\infty} u_k'(x) = \sum_{k=1}^{\infty} \left(\frac{-k \sin kx}{x^2 + k^3} + \frac{-2x \cos kx}{(x^2 + k^3)^2} \right)$$

is uniformly convergent. We see that

$$\left| \frac{-k \sin kx}{x^2 + k^3} + \frac{-2x \cos kx}{(x^2 + k^3)^2} \right| \leq \frac{k}{x^2 + k^3} + \frac{2|x|}{(x^2 + k^3)^2} \leq \frac{1}{k^2} + \frac{2|x|}{(x^2 + k^3)^2}.$$

Furthermore,

$$|x| \leq k \quad \Rightarrow \quad \frac{2|x|}{(x^2 + k^3)^2} \leq \frac{2k}{k^6} = \frac{2}{k^5}$$

and

$$|x| > k \quad \Rightarrow \quad \frac{2|x|}{(x^2 + k^3)^2} \leq \frac{2|x|}{|x|^4} = \frac{2}{|x|^3} \leq \frac{2}{k^3},$$

so it is clear that $v(x) = \sum_{k=1}^{\infty} u_k'(x)$ is uniformly convergent by the M-test. Since all u_k' are continuous, it follows that v is continuous. The fact that $u(x) = \sum_{k=1}^{\infty} u_k(x)$ is convergent and that the series defining the continuous function v is uniformly convergent, we obtain that $u'(x) = v(x)$, which proves that $u \in C^1(\mathbf{R})$.

Answer: See above.