Transform theory 2025-01-08 – Solutions

1. (a) Yes. Since $u_n(x) \to 0$ for $|x| \le 1/4$ and $|u_n(x) - 0| = |x|^n \le 4^{-n} \to 0$ independent of $x \in [-1/4, 1/4]$.

   (b) No, an exponentially bounded function has a convergent Laplace transform for all $\operatorname{Re} s > a$, where a is some real number. Moreover, the Laplace transform is analytic there (so continuous). Hence the described behavior is not possible.

   (c) No. For example $u[k] = v[k] = 1$. Then $\mathcal{Z}(u) + \mathcal{Z}(v) = 2z/(z-1)$. However, it is clear that $\mathcal{Z}(uv) = z/(z-1)$.

   (d) No. Not necessarily. The Fourier transform is bounded, but does not need to belong to $L^1(\mathbf{R})$. There's no such result. A counter example is a bit tricky though.

   (e) Yes. Note that $\cos\frac{k\pi}{4}\sin\frac{k\pi}{4} = \frac{1}{2}\sin\frac{k\pi}{2}$ and that $\mathcal{Z}(\sin\frac{k\pi}{2}) = \frac{z}{z^2+1}$, $|z| > 0$.

   Answer: Yes. No. No. No. Yes.

2. We assume that y, y', y'' all belong to X_a (and verify this at the end). Taking the Laplace transform of the equation, we obtain that

$$s^2Y(s) - sy(0) - y'(0) + 2\big(sY(s) - y(0)\big) - 8Y(s) = \frac{-9}{(s+1)^2}$$
$$\Leftrightarrow \quad Y(s)\left(s^2 + 2s - 8\right) = 2s + 3 + \frac{-9}{(s+1)^2}, \quad \operatorname{Re} s > -1.$$

   Hence, since $s^2 + 2s - 8 = (s+4)(s-2)$,

$$Y(s) = \frac{-9}{(s+4)(s+1)^2(s-2)} + \frac{2s+3}{(s+4)(s-2)} = \frac{2s^3 + 7s^2 + 8s - 6}{(s+4)(s+1)^2(s-2)}$$
$$= \frac{1}{s+4} + \frac{1}{(s+1)^2} + \frac{1}{s-2},$$

   after decomposing into partial fractions. Note that the term $A/(s+1)$ yields $A = 0$. Not obvious. From a table,

$$\mathcal{L}(e^{-4t}) = \frac{1}{s+4}, \quad \mathcal{L}(e^{2t}) = \frac{1}{s-2}, \quad \text{and} \quad \mathcal{L}(te^{-t}) = \frac{1}{(s+1)^2},$$

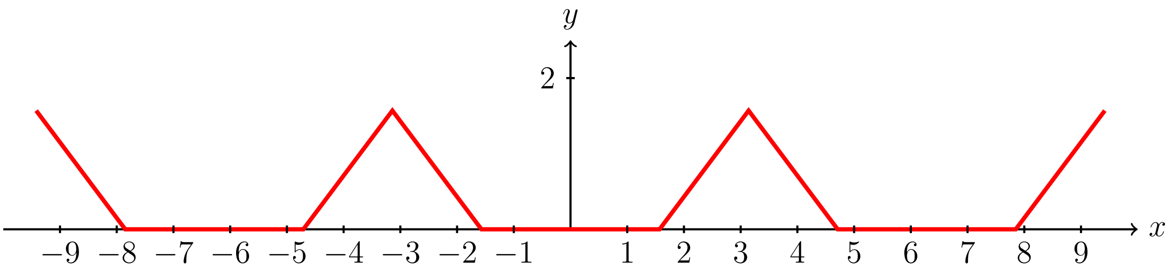
   for $\operatorname{Re} s > 2$. Hence

$$y(t) = e^{-4t} + te^{-t} + e^{2t}$$

   by uniqueness and linearity. Obviously y and its derivatives are exponentially bounded.

   Answer: $y(t) = e^{-4t} + te^{-t} + e^{2t}, \quad t > 0$.

3. Clearly $u \in E$. This is obvious since the the periodic extension function is continuous everywhere. Furthermore, u is infinitely differentiable for $x \ne n\pi/2$ (actually $x = 2m\pi$ is OK), and at $x = n\pi/2$ the right- and lefthand derivatives exist. Hence – by Dirichlet's theorem – the Fourier series of u is convergent and converges to $u(x)$ for all $x \in \mathbf{R}$. Moreover, since u' is piecewise constant, it is clear that $u' \in E$. We also have $u(-\pi) = u(\pi)$ so the convergence of the Fourier series is uniform by theorem (see Lecture 4). We sketch the graph of the Fourier series (which in this case is equal to $u(x)$) below.



Note that $u(x)$ is an even function, so $b_k = 0$ for all k. For $k > 0$, again using the fact that u is even,

$$
\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\cos kx\,dx = \frac{2}{\pi}\int_{\pi/2}^{\pi}\left(x-\frac{\pi}{2}\right)\cos kx\,dx \\
&= \frac{2}{\pi}\left(\left[\frac{\left(x-\frac{\pi}{2}\right)\sin kx}{k}\right]_{\pi/2}^{\pi} - \frac{1}{k}\int_{\pi/2}^{\pi}\sin kx\,dx\right) \\
&= \frac{2}{\pi}\left(0-0+\frac{1}{k^2}\left[\cos kx\right]_{\pi/2}^{\pi}\right) = \frac{2\cos k\pi - 2\cos\frac{k\pi}{2}}{\pi k^2} = \frac{2\big((-1)^k - \cos\frac{k\pi}{2}\big)}{\pi k^2}
\end{aligned}
$$

and

$$a_0 = \frac{2}{\pi}\int_{\pi/2}^{\pi}\left(x-\frac{\pi}{2}\right)\,dx = \frac{\pi}{4}.$$

Hence

$$u(x) \sim \frac{\pi}{8} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k - \cos\frac{k\pi}{2}}{k^2}\cos kx.$$

Answer: $u(x) \sim \dfrac{\pi}{8} + \dfrac{2}{\pi}\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k - \cos\frac{k\pi}{2}}{k^2}\cos kx$; see above.

4. Assuming that $y, y', y'' \in G$, we take the Fourier transform to find that

$$(i\omega)^2 Y(\omega) - Y(\omega) = \frac{3}{4}\mathcal{F}(e^{-2|x-1|}) = \frac{3}{4}e^{-i\omega}\frac{2\cdot 2}{2^2+\omega^2},$$

where we used the rule $\mathcal{F}(y(x-1)) = e^{-i\omega}\mathcal{F}(y(x))$ and a table. Hence

$$Y(\omega) = \frac{-3e^{-i\omega}}{(1+\omega^2)(4+\omega^2)} = e^{-i\omega}\left(\frac{-1}{\omega^2+1} + \frac{1}{4+\omega^2}\right),$$

after decomposing into partial fractions (let $\xi = \omega^2$ to make this step easier). From a table, we again find that

$$\mathcal{F}(e^{-|x|}) = \frac{2}{1+\omega^2} \quad\text{and}\quad \mathcal{F}(e^{-2|x|}) = \frac{4}{4+\omega^2},$$

so by the rule $\mathcal{F}(y(x-1)) = e^{-i\omega}\mathcal{F}(y(x))$ (again), uniqueness and linearity,

$$y(x) = -\frac{1}{2}e^{-|x-1|} + \frac{1}{4}e^{-2|x-1|}.$$

This function and its piecewise derivatives up to order 2 are absolutely integrable.

Answer: $y(x) = -\dfrac{1}{2}e^{-|x-1|} + \dfrac{1}{4}e^{-2|x-1|}$.

5. The sum in the equation is the convolution of u with the function $k \mapsto k$ (for $k \geq 0$). We take the Z transform of the equation and find that

$$U(z)\frac{z}{(z-1)^2} = 2 \cdot \frac{z}{z-2} - 2 \cdot \frac{z}{z+1},$$

where we assume that (at least) $|z| > 2$. Reformulating this equation, we find that

$$U(z) = \frac{(z-1)^2}{z}\left(2 \cdot \frac{z}{z-2} - 2 \cdot \frac{z}{z+1}\right) = \frac{6(z-1)^2}{(z-2)(z+1)}.$$

Rewriting the right-hand side and decomposing into partial fractions yields

$$\begin{aligned} z \cdot \frac{6z-12}{(z-2)(z+1)} + \frac{6}{(z-2)(z+1)} &= z \cdot \frac{6}{z+1} - \frac{2}{z+1} + \frac{2}{z-2} \\ &= \frac{6z}{z+1} + \frac{1}{z}\left(-\frac{2z}{z+1} + \frac{2z}{z-2}\right). \end{aligned}$$

Using the rule $\mathcal{Z}(u[n-1]H[n-1]) = z^{-1}\,\mathcal{Z}(u[n])$ and the identities

$$\mathcal{Z}\left((-1)^n\right) = \frac{z}{z+1} \quad \text{and} \quad \mathcal{Z}\left(2^n\right) = \frac{z}{z-2},$$

we obtain by linearity and uniqueness that

$$u[n] = 6(-1)^n + H[n-1]\left(-2(-1)^{n-1} + 2 \cdot 2^{n-1}\right) = 6(-1)^n + H[n-1]\left(2(-1)^n + 2^n\right).$$

Answer: $u[n] = 6(-1)^n + H[n-1]\left(2(-1)^n + 2^n\right)$, $k = 0, 1, 2, 3, \ldots$.

6. We note that by dividing the equation by 2π, the lefthand side is the periodic convolution of u and the periodic function defined by $v(\tau) = e^\tau$ for $0 \leq \tau < 2\pi$. How we choose to define this function at the endpoints does not matter since we're under the integral sign. So we're solving

$$\frac{1}{2\pi}\int_0^{2\pi} v(\tau)u(t-\tau)\,d\tau = \frac{1}{2\pi}\cos 2t = \frac{1}{4\pi}e^{i2t} + \frac{1}{4\pi}e^{-i2t}.$$

Since the left-hand side is 2π-periodic, it has Fourier coefficients and from a table we find that

$$\widehat{v}[k]\,\widehat{u}[k] = \begin{cases} \frac{1}{4\pi}, & k = \pm 2, \\ 0, & k \neq \pm 2. \end{cases} \tag{$\dagger$}$$

We need to find $\widehat{v}[k]$:

$$\widehat{v}[k] = \frac{1}{2\pi}\int_0^{2\pi} e^\tau e^{-ik\tau}\,d\tau = \frac{1}{2\pi}\int_0^{2\pi} e^{(1-ik)\tau}\,d\tau = \frac{1}{2\pi}\left[\frac{e^{(1-ik)\tau}}{1-ik}\right]_0^{2\pi} = \frac{e^{2\pi}-1}{2\pi(1-ik)}.$$

In particular we see that $\widehat{v}[k] \neq 0$ for all k. By equation ($\dagger$), this means that $\widehat{u}[k] = 0$ for $k \neq \pm 2$. For $k = \pm 2$, we find that

$$\widehat{u}[k] = \frac{1}{4\pi}\,\frac{2\pi(1-ik)}{e^{2\pi}-1} = \frac{1}{2}\,\frac{1-ik}{e^{2\pi}-1}.$$

Hence

$$u(t) \sim \sum_{k\in\mathbf{Z}} \widehat{u}[k]e^{ikt} = \frac{1}{2(e^{2\pi}-1)}\big((1+2i)e^{-i2t} + (1-2i)e^{i2t}\big)$$
$$= \frac{1}{2(e^{2\pi}-1)}\left(e^{-i2t} + e^{i2t} + 4\,\frac{e^{-i2t}-e^{i2t}}{2}\right) = \frac{1}{e^{2\pi}-1}\left(\cos 2t + 2\sin 2t\right).$$

Answer: $u(x) = \dfrac{1}{e^{2\pi}-1}(\cos 2t + 2\sin 2t)$.

Alternate solution: Suppose that u can be expressed as a uniformly convergent Fourier series: $u(t) = \sum_{k\in\mathbf{Z}} c_k e^{ikt}$. It is not obvious that this is possible, so we need to verify when (and if) we find a solution. Plugging this series into the equation we find

$$\int_0^{2\pi} e^{\tau}u(t-\tau)\,d\tau = \int_0^{2\pi} e^{\tau}\sum_{k\in\mathbf{Z}} c_k e^{ik(t-\tau)}\,d\tau = \sum_{k\in\mathbf{Z}} c_k \int_0^{2\pi} e^{\tau}\,e^{ik(t-\tau)}\,d\tau$$
$$= \sum_{k\in\mathbf{Z}} c_k e^{ikt}\left[\frac{e^{\tau(1-ik)}}{1-ik}\right]_0^{2\pi} = \sum_{k\in\mathbf{Z}} \frac{c_k(e^{2\pi}-1)}{1-ik}e^{ikt} = \cos 2t = \frac{1}{2}e^{-i2t} + \frac{1}{2}e^{i2t},$$

where changing the order of summation and integration is motivated by the uniform convergence. Thus, for $k \neq \pm 2$, we must have

$$\frac{c_k(e^{2\pi}-1)}{1-ik} = 0 \quad \Leftrightarrow \quad c_k = 0.$$

For $k = \pm 2$,

$$\frac{c_k(e^{2\pi}-1)}{1-ik} = \frac{1}{2} \quad \Leftrightarrow \quad c_k = \frac{1-ik}{2(e^{2\pi}-1)}.$$

This leads to the same function found with the previous method: $u(t) = \dfrac{\cos 2t + 2\sin 2t}{e^{2\pi}-1}$. Obviously this function has a Fourier series that converges uniformly to u, which motivates the assumptions above.

7. Let $u_n(x) = \dfrac{(n^2+1)\sin x + n\cos x + 2}{1 + n\sin x + n^2\cos x}$. We find the pointwise limit as $n \to \infty$:

$$u_n(x) = \frac{(1+1/n^2)\sin x + (1/n)\cos x + 2/n^2}{1/n^2 + (1/n)\sin x + \cos x} \to \frac{\sin x}{\cos x}.$$

If the convergence is uniform, we can move the limit inside the integral. We prove that the convergence is uniform on $[0, \pi/4]$. We see that

$$\left|u_n(x) - \frac{\sin x}{\cos x}\right| = \left|\frac{\big((n^2+1)\sin x + n\cos x + 2\big)\cos x - \big(1 + n\sin x + n^2\cos x\big)\sin x}{(1 + n\sin x + n^2\cos x)\cos x}\right|$$
$$= \left|\frac{n(\cos^2 x - \sin^2 x) + \cos x\sin x + 2\cos x - \sin x}{(1 + n\sin x + n^2\cos x)\cos x}\right|$$
$$\leq \frac{2n+4}{|1 + n\sin x + n^2\cos x||\cos x|} = \frac{2/n + 4/n^2}{|1/n^2 + (1/n)\sin x + \cos x||\cos x|}.$$

For $0 \leq x \leq \pi/4$, we know that $\sin x \geq 0$ and $\cos x \geq \dfrac{\sqrt{2}}{2}$. Thus

$$\left|\frac{1}{n^2} + \frac{1}{n}\sin x + \cos x\right| |\cos x| \geq \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2},$$

so

$$\frac{2/n + 4/n^2}{|1/n^2 + (1/n)\sin x + \cos x||\cos x|} \leq 2\left(\frac{2}{n} + \frac{4}{n^2}\right) \to 0, \quad \text{as } n \to \infty,$$

independently of x. Therefore the convergence is uniform on $[0, \pi/4]$ and

$$\lim_{n\to\infty} \int_0^{\pi/4} u_n(x)\,dx = \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx = \big[-\ln\cos x\big]_0^{\pi/4} = -\ln\cos\frac{\pi}{4} + \ln\cos 0 = \frac{\ln 2}{2}.$$

Answer: $\dfrac{\ln 2}{2}$.

Note: The bit about careful motivation is due to the problem of moving the limit inside the integral. We show that this is allowed by proving that the sequence is uniformly convergent on the interval we integrate over.