Transform theory 2025-06-05 – Solutions

1. (a) Yes. Since $u_n(x) \to 0$ for $|x| \le 1/2$ and $|u_n(x) - 0| = |x|^{3n} \le 2^{-3n} \to 0$ independent of $x \in [-1/2, 1/2]$.

   (b) Yes. The function is continuous (so piecewise continuous) and the area beneath the graph is equal to one: $\displaystyle\sum_{k=0}^{\infty} 2^{-k-1} = 1$, so the functions belongs to $G(\mathbf{R})$.

   (c) Yes. The partial sums are continuous functions so uniform convergence ensures that the limiting function is continuous.

   (d) Yes. This is clear because

$$|\,\mathcal{L}\,u(s)| = \left|\int_0^{\infty} u(t)\,e^{-st}\,dt\right| \le \int_0^{\infty} |u(t)|\,e^{-t\,\mathrm{Re}\,s}\,dt \le \int_0^{\infty} |u(t)|\,dt < \infty$$

   since $t\,\mathrm{Re}\,s \ge 0$. Note that this is possible even if u does not belong to X_a for any $a > 0$.

   (e) Yes. This is clear since

$$\sum_{k=0}^{\infty} \left|u[k]z^{-k}\right| \le C \sum_{k=0}^{\infty} |z|^{-k} = \frac{C}{1 - |z|^{-1}} < \infty$$

   if $|z| > 1$, so the series is absolutely convergent.

Answer: Yes. Yes. Yes. Yes. Yes. (yes..)

2. Assuming that $y, y', y'' \in G$, we take the Fourier transform to find that

$$(i\omega)^2 Y(\omega) - 4Y(\omega) = 36\,\mathcal{F}(xe^x H(-x)) = 36i\frac{d}{d\omega}\,\mathcal{F}(e^x H(-x)) = \frac{-36}{(1 - i\omega)^2}.$$

Hence, by decomposing into partial fractions,

$$Y(\omega) = \frac{-36}{((i\omega)^2 - 4)(i\omega - 1)^2} = \frac{1}{i\omega + 2} + \frac{8}{i\omega - 1} - \frac{9}{i\omega - 2} + \frac{12}{(i\omega - 1)^2}$$

From a table, we find that

$$\mathcal{F}(e^{-2x}H(x)) = \frac{1}{2 + i\omega}, \quad \mathcal{F}(e^{2x}H(-x)) = \frac{1}{2 - i\omega}, \quad \mathcal{F}(e^{x}H(-x)) = \frac{1}{1 - i\omega},$$
$$\text{and} \quad \mathcal{F}(xe^x H(-x)) = i\frac{d}{d\omega}\,\mathcal{F}(e^x H(-x)) = \frac{-1}{(1 - i\omega)^2},$$

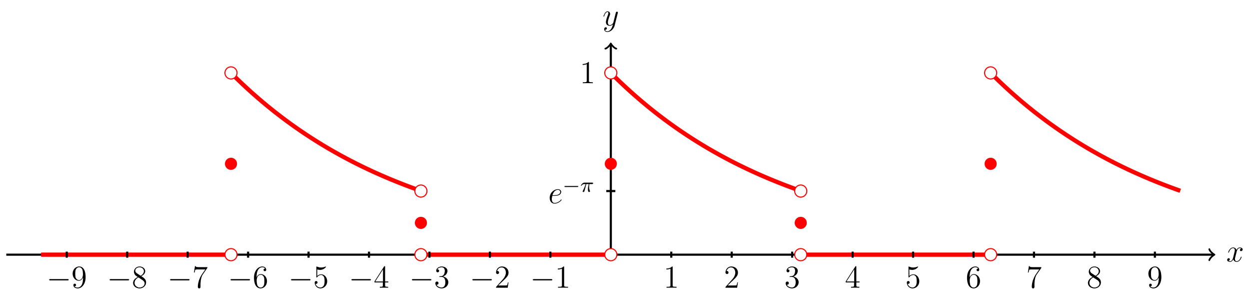
so by uniqueness and linearity,

$$y(x) = e^{-2x}H(x) - (8 + 12x)e^x H(-x) + 9e^{2x}H(-x).$$

This function and its piecewise derivatives up to order 2 are absolutely integrable.

Answer: $y(x) = \begin{cases} 9e^{2x} - (8 + 12x)e^x, & x < 0, \\ e^{-2x}, & x > 0. \end{cases}$

3. Clearly $u \in E$. This is obvious since the the periodic extension function is continuous for $k\pi < x < (k+1)\pi$ and has one-sided limits at $x = k\pi$. Furthermore, u is infinitely differentiable for $x \neq n\pi$, and at $x = n\pi$ the right- and lefthand derivatives exist. Hence – by Dirichlet's theorem – the Fourier series of u is convergent and converges to $u(x)$ for all $x \neq k\pi$. For $x = (2k+1)\pi$ we get $(u(x^+) + u(x^-))/2 = (e^{-\pi} + 0)/2 = e^{-\pi}/2$ and for $x = 2k\pi$ we get $(u(x^+) + u(x^-))/2 = (e^0 + 0)/2 = 1/2$; see the figure below. Since the graph (depicting the Fourier series) has discontinuities, the convergence can not be uniform.



We find that, for all $k \in \mathbf{Z}$,

$$
\begin{aligned}
c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx = \frac{1}{2\pi}\int_0^{\pi} e^{-x}e^{-ikx}\,dx = \frac{1}{2\pi}\int_0^{\pi} e^{-(1+ik)x}\,dx = \frac{1}{2}\left[\frac{e^{-ikx}}{-(1+ik)}\right]_0^{\pi} \\
&= \frac{1}{2(1+ik)}\left(1 - e^{-(1+ik)\pi}\right) = \frac{1-(-1)^k e^{-\pi}}{2\pi(1+ik)}.
\end{aligned}
$$

Hence

$$
u(x) \sim \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \frac{1-(-1)^k e^{-\pi}}{1+ik}\, e^{ikx}.
$$

Notice that

$$
|c_k|^2 = \left|\frac{1-(-1)^k e^{-\pi}}{2\pi(1+ik)}\right|^2 = \frac{(1-(-1)^k e^{-\pi})^2}{(2\pi)^2|1+ik|^2} = \frac{(1-(-1)^k e^{-\pi})^2}{4\pi^2(1+k^2)}
$$

so by Parsevals's formula we obtain that

$$
\sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|^2\,dx = \frac{1}{2\pi}\int_0^{\pi} e^{-2x}\,dx = \frac{1}{2\pi}\left[\frac{e^{-2x}}{-2}\right]_0^{\pi} = \frac{1}{4\pi}\left(1 - e^{-2\pi}\right)
$$

which we can reformulate as

$$
\sum_{k=-\infty}^{\infty} \frac{(1-(-1)^k e^{-\pi})^2}{4\pi^2(1+k^2)} = \frac{1}{4\pi}\left(1 - e^{-2\pi}\right) \quad \Leftrightarrow \quad \sum_{k=-\infty}^{\infty} \frac{(1-(-1)^k e^{-\pi})^2}{1+k^2} = \pi\left(1 - e^{-2\pi}\right).
$$

Answer: $u(x) \sim \dfrac{1}{2\pi}\displaystyle\sum_{k=-\infty}^{\infty} \frac{1-(-1)^k e^{-\pi}}{1+ik}\, e^{ikx}$; $\pi\left(1 - e^{-2\pi}\right)$.

4. We take the Z transform of the equation and find that

$$z^2U(z) - z^2u[0] - zu[1] - \big(zU(z) - zu[0]\big) - 6U(z) = 4 \cdot \frac{z}{z-2} + 5 \cdot \frac{z}{z+2},$$

where we assume that (at least) $|z| > 2$. Reformulating this equation, we find that

$$\begin{aligned}(z^2 - z - 6)U(z) &= 5z^2 + \frac{4z}{z-2} + \frac{5z}{z+2} = \frac{z(5z^3 - 11z - 2)}{z^2 - 4}\\ \Leftrightarrow \quad \frac{U(z)}{z} &= \frac{5z^3 - 11z - 2}{(z+2)^2(z-2)(z-3)}.\end{aligned}$$

Rewriting the right-hand side and decomposing into partial fractions yields

$$\frac{U(z)}{z} = \frac{4}{z-3} + \frac{2}{z+2} - \frac{1}{(z+2)^2} - \frac{1}{z-2}.$$

Using the identities

$$\begin{aligned}\mathcal{Z}\left(2^k\right) = \frac{z}{z-2}, \quad \mathcal{Z}\left(3^k\right) = \frac{z}{z-3}, \quad \mathcal{Z}\left((-2)^k\right) = \frac{z}{z+2}\\ \text{and} \quad \mathcal{Z}\left(k(-2)^k\right) = \frac{-2z}{(z+2)^2},\end{aligned}$$

we obtain by linearity and uniqueness that

$$u[k] = 4 \cdot 3^k + 2 \cdot (-2)^k + \frac{k}{2} \cdot (-2)^k - 2^n = 4 \cdot 3^k - 2^k + \left(2 + \frac{k}{2}\right)(-2)^k.$$

Answer: $u[k] = 4 \cdot 3^k - 2^k + \left(2 + \frac{k}{2}\right)(-2)^k$, $k = 0, 1, 2, 3, \ldots$.

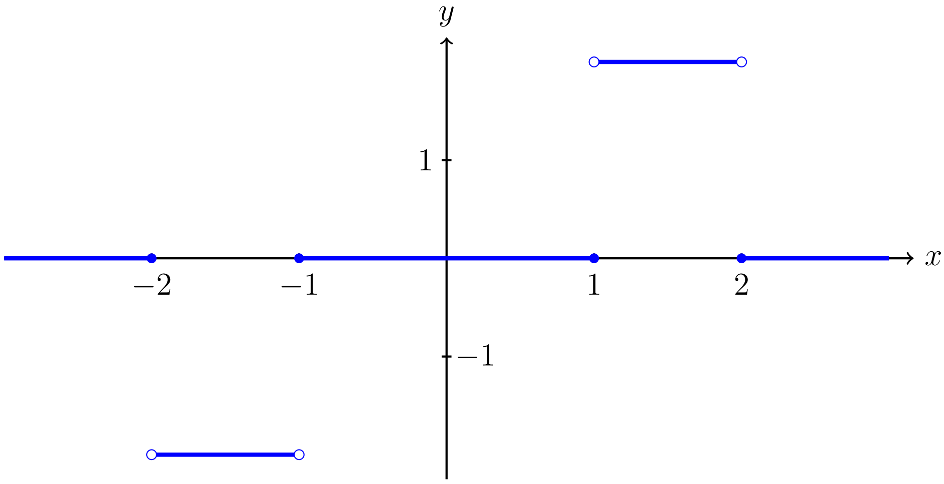
5. (a) We observe that $f \in G(\mathbf{R})$ so the Fourier transform exists and

$$\begin{aligned}F(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \int_{-2}^{-1} -2\,e^{-i\omega x}\,dx + \int_{1}^{2} 2\,e^{-i\omega x}\,dx\\ &= \left[-\frac{2\,e^{-i\omega x}}{-i\omega}\right]_{-2}^{-1} + \left[\frac{2\,e^{-i\omega x}}{-i\omega}\right]_{1}^{2} = \frac{4}{-i\omega}\left(\frac{e^{i2\omega} - e^{i\omega}}{2} + \frac{e^{-i2\omega} - e^{-i\omega}}{2}\right)\\ &= \frac{4i}{\omega}\left(\cos 2\omega - \cos\omega\right), \quad \omega \neq 0.\end{aligned}$$

At $\omega = 0$, we calculate directly:

$$F(0) = \int_{-\infty}^{\infty} f(x)e^{-i\cdot 0\cdot x}\,dx = 0$$

since $f \in L^1(\mathbf{R})$ is odd. Drawing the graph of $f(x)$, we find the following.



(b) Notice that

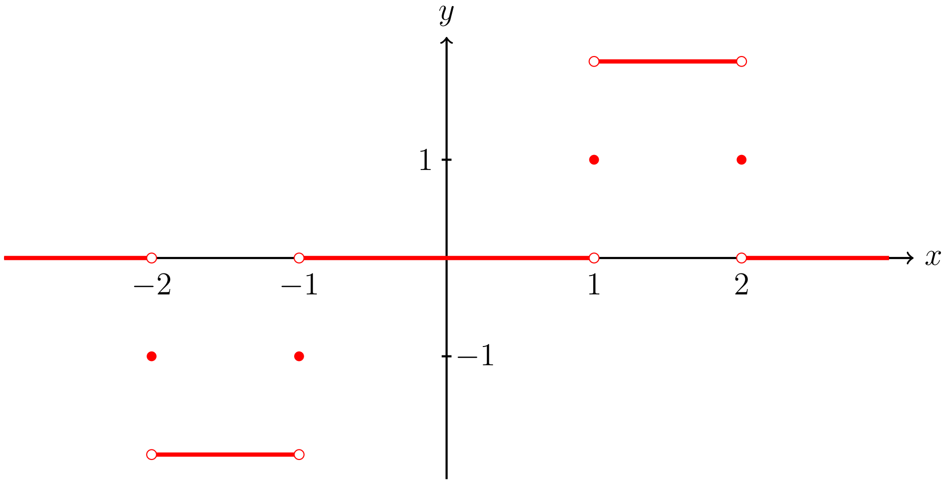
$$\frac{\cos\omega - \cos 2\omega}{\omega} = \frac{1}{4i}\cdot\frac{4i}{\omega}(\cos\omega - \cos 2\omega) = \frac{i}{4}F(\omega).$$

Since $D^{\pm}f(x)$ exists for every $x \in \mathbf{R}$, Fourier inversion (Dirichlet's theorem for the Fourier transform) yields

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{\cos\omega - \cos 2\omega}{\omega}\,e^{i\omega x}\,d\omega = \frac{2\pi i}{4}\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R}F(\omega)\,e^{i\omega x}\,d\omega$$

$$= \frac{i\pi}{2}\cdot\frac{f(x^+)+f(x^-)}{2} = \begin{cases} -i\pi, & -2<x<-1,\\ i\pi, & 1<x<2,\\ \pm i\pi/2, & x=\pm 2, \pm 1,\\ 0, & |x|>2 \text{ or } |x|<1.\end{cases}$$

We can draw the graph for the function defined by $\mathcal{F}^{-1}(F(\omega))(x)$.



Rewriting the integrand, we find that

$$\frac{(\cos\omega - \cos 2\omega)\sin\omega}{\omega} = \frac{(\cos\omega - \cos 2\omega)\operatorname{Im} e^{i\omega}}{\omega} = \operatorname{Im}\frac{(\cos\omega - \cos 2\omega)e^{i\omega}}{\omega}$$

so by the previous result (with $x = 1$),

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{(\cos\omega - \cos 2\omega)\sin\omega}{\omega}\,d\omega = \operatorname{Im}\left(\frac{i\pi}{2}\right) = \frac{\pi}{2}.$$

Answer: (a) $F(\omega) = \dfrac{4i(\cos 2\omega - \cos\omega)}{\omega}$, $\omega \neq 0$, $F(0) = 0$ (b) see above.

6. (a) Easiest is to write the RHS as

$$f(t) = \sin t - \sin t H(t-\pi) = \sin t + \sin(t-\pi)H(t-\pi)$$

and use $\mathcal{L}(u(t-a)H(t-a)) = e^{-as}\,\mathcal{L}(u)$ with $u(t) = \sin t$ to obtain

$$F(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}$$

for $\operatorname{Re} s > 0$. Alternatively, it is not an unreasonable approach to use direct integration:

$$\begin{aligned}
F(s) &= \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^\pi \sin t\,e^{-st}\,dt = \frac{1}{2i}\int_0^\pi \left(e^{it} - e^{-it}\right)e^{-st}\,dt \\
&= \frac{1}{2i}\left[\frac{e^{(i-s)t}}{i-s} + \frac{e^{-(i+s)t}}{i+s}\right]_0^\pi = \frac{1}{2i}\left(e^{-s\pi}\left(\frac{e^{i\pi}}{i-s} + \frac{e^{-i\pi}}{i+s}\right) - \left(\frac{1}{i-s} + \frac{1}{i+s}\right)\right) \\
&= \frac{-1}{2i}\left(e^{-s\pi}\left(\frac{1}{i-s} + \frac{1}{i+s}\right) + \left(\frac{1}{i-s} + \frac{1}{i+s}\right)\right) \\
&= \frac{-(1+e^{-\pi s})}{2i}\left(\frac{1}{i-s} + \frac{1}{i+s}\right) = \frac{1+e^{-\pi s}}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) \\
&= \frac{1+e^{-\pi s}}{2i}\cdot\frac{2i}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1},
\end{aligned}$$

with $\operatorname{Re} s > 0$ to avoid $s = \pm i$.

(b) The integral in the left-hand side is the one-sided convolution of u with $t \mapsto e^{2t}$, so taking the Laplace transform yields

$$sU(s) - u(0) + U(s)\,\mathcal{L}(e^{2t}) = \frac{1+e^{-\pi s}}{s^2+1}, \quad \operatorname{Re} s > 0.$$

Thus

$$U(s)\left(s + \frac{1}{s+2}\right) = \frac{1+e^{-\pi s}}{s^2+1} \quad \Leftrightarrow \quad U(s) = \left(1+e^{-\pi s}\right)\frac{s-2}{(s-1)^2(s^2+1)},$$

for $\operatorname{Re} s > 1$. Decomposing into partial fractions we find that

$$\begin{aligned}
\frac{s-2}{(s-1)^2(s^2+1)} &= \frac{-s}{s^2+1} - \frac{1/2}{s^2+1} + \frac{1}{s-1} - \frac{1/2}{(s-1)^2} \\
&= \mathcal{L}\left(-\cos t - \frac{1}{2}\sin t + \frac{1}{2}(2-t)\,e^t\right).
\end{aligned}$$

Since $\mathcal{L}(v(t-\pi)H(t-\pi)) = e^{-\pi s}(\mathcal{L}\,v)(s)$, we find by uniqueness that

$$\begin{aligned}
u(t) &= -\cos t - \frac{1}{2}\sin t + \frac{1}{2}(2-t)\,e^t \\
&\quad + \left(-\cos(t-\pi) - \frac{1}{2}\sin(t-\pi) + \frac{1}{2}(2-(t-\pi))\,e^{t-\pi}\right)H(t-\pi) \\
&= -\cos t - \frac{1}{2}\sin t + \frac{1}{2}(2-t)\,e^t \\
&\quad + \left(\cos t + \frac{1}{2}\sin t + \frac{1}{2}(2-(t-\pi))\,e^{t-\pi}\right)H(t-\pi) \\
&= \begin{cases} -\cos t - \frac{1}{2}\sin t + \frac{1}{2}(2-t)\,e^t, & 0 < t < \pi, \\ \frac{1}{2}(2-t)\,e^t + \frac{1}{2}(2-(t-\pi))\,e^{t-\pi}, & t > \pi. \end{cases}
\end{aligned}$$

Answer: (a) $F(s) = \dfrac{1+e^{-\pi s}}{s^2+1}$, $\operatorname{Re} s > 0$ (b) $\begin{cases} -\cos t - \dfrac{1}{2}\sin t + \dfrac{1}{2}(2-t)\,e^t, & 0 < t < \pi, \\ \dfrac{e^t}{2}\left(2 - t + (2-(t-\pi))\,e^{-\pi}\right), & t > \pi. \end{cases}$

7. Since

$$0 \le \frac{1}{(x+k)^{3/2}} \le \frac{1}{k^{3/2}}, \quad x \ge 0,\ k = 1, 2, 3, \ldots,$$

and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is convergent, it follows from Weierstrass M-test that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(x+k)^{3/2}}$ is uniformly convergent on $[0,1]$. Thus we can change the order of integration and summation, yielding that

$$\begin{aligned}
\int_0^1 \left(\sum_{k=1}^{\infty} \frac{1}{(x+k)^{3/2}} \right) dx &= \sum_{k=1}^{\infty} \int_0^1 \frac{1}{(x+k)^{3/2}}\,dx = \sum_{k=1}^{\infty} \left[-\frac{2}{(x+k)^{1/2}} \right]_0^1 \\
&= 2\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right) = \lim_{n\to\infty} 2\sum_{k=1}^{n} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right) \\
&= \lim_{n\to\infty} 2\left(1 - \frac{1}{\sqrt{n+1}} \right) = 2
\end{aligned}$$

since the integrated series is a telescoping sum (write out a couple of terms to ensure this!).

Answer: 2.

Note: The bit about careful motivation is due to the problem of moving the limit inside the integral. We show that this is allowed by proving that the sequence is uniformly convergent on the interval we integrate over.