Transform theory 2025-08-21 – Solutions

1. (a) No. Since $u_n(x) \to 0$ for $|x| < 1$ while $u_n(\pm 1) = 1$ for all n, it is clear that the limiting function is discontinuous. Therefore the convergence can not be uniform.

   (b) No. The function described in the graph does not tend to zero as $\omega \to \infty$, therefore violating the Riemann-Lebesgue lemma.

   (c) Yes. The partial sums are continuous functions and absolute convergence for a Fourier series implies uniform convergence, so this ensures that the limiting function is continuous.

   (d) No. This is clear because for instance $t \mapsto e^{-t/4}$ belongs to $L^1(0,\infty)$ but with $s = -1/2$ (so $\operatorname{Re} s > -1$) we find that

$$\mathcal{L}\, u(-1/2) = \int_0^\infty u(t)\, e^{-(-1/2)t}\, dt = \int_0^\infty e^{-t/4}\, e^{t/2}\, dt = \int_0^\infty e^{t/4}\, dt = \infty,$$

   so the Laplace transform does not exist for $s = -1/2$ (or any s such that $\operatorname{Re} s \leq -1/4$).

   (e) Yes. This is clear since

$$\frac{1}{1 - 3/z} = \frac{z}{z-3}$$

   which is the Z transform of 3^k for $|z| > 3$ (table).

Answer: No. No. Yes. No. Yes.

2. (a) Taking the Z transform with $|z| > 1$ yields

$$\begin{aligned} z^2 U(z) - z^2 u[0] - z u[1] - (zU(z) - zu[0]) - 2U(z) &= \frac{6z}{z-1} \\ \Leftrightarrow \quad \left(z^2 - z - 2\right) U(z) &= z^2 + z + \frac{6z}{z-1}. \end{aligned}$$

   Thus, since $z^2 - z - 2 = (z-2)(z+1)$,

$$\begin{aligned} U(z) &= \frac{z}{z-2} + z \cdot \frac{6}{(z-1)(z-2)(z+1)} = \frac{z}{z-2} + z\left(\frac{1}{z+1} - \frac{3}{z-1} + \frac{2}{z-2}\right) \\ &= \frac{3z}{z-2} + \frac{z}{z+1} - \frac{3}{z-1}, \end{aligned}$$

   where we decomposed into partial fractions. We can now use a table (and uniqueness) to find that

$$u[k] = (-1)^k + 3\left(2^k - 1\right).$$

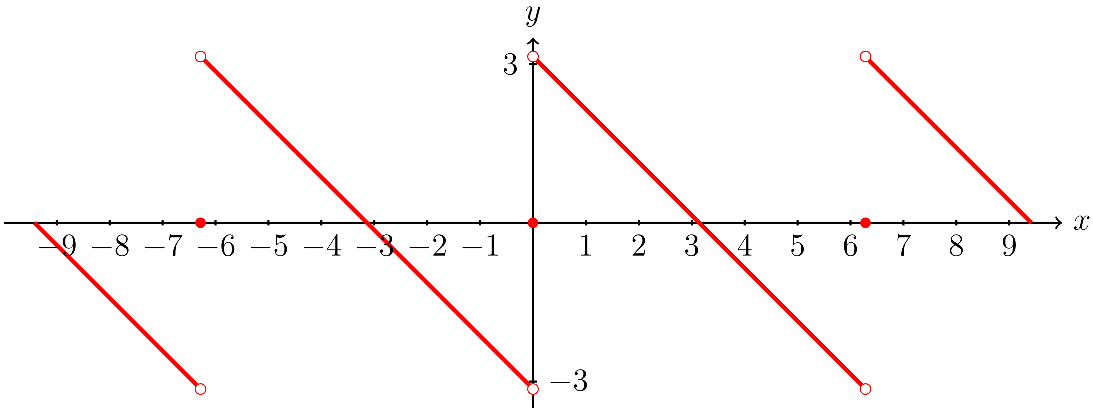
   (b) Using Euler's equations, we find that

$$\begin{aligned} \mathcal{Z}\left(\frac{e^{ik\alpha} + e^{-ik\alpha}}{2}\right) &= \frac{1}{2}\left(\frac{z}{z - e^{i\alpha}} + \frac{z}{z - e^{-i\alpha}}\right) = \frac{1}{2}\left(\frac{z(z - e^{-i\alpha}) + z(z - e^{i\alpha})}{(z - e^{i\alpha})(z - e^{-i\alpha})}\right) \\ &= \frac{1}{2}\left(\frac{2z^2 - z(e^{-i\alpha} + e^{i\alpha})}{z^2 - z(e^{i\alpha} + e^{-i\alpha}) + 1}\right) = \frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1}, \end{aligned}$$

   since $\mathcal{Z}(a^k) = z/(z-a)$ for $a \in \mathbf{C}$ $(a \neq 0)$.

Answer: $u[k] = (-1)^k + 3\left(2^k - 1\right)$, $k = 0, 1, 2, 3, \ldots$; see above.

3. Clearly $u \in E$. This is obvious since the the periodic extension function is continuous for $2k\pi < x < (2k+2)\pi$ and has one-sided limits at $x = 2k\pi$. Furthermore, u is infinitely differentiable for $x \neq 2k\pi$, and at $x = 2k\pi$ the right- and lefthand derivatives exist. Hence – by Dirichlet's theorem – the Fourier series of u is convergent and converges to $u(x)$ for all $x \neq 2k\pi$. For $x = 2k\pi$ we get $(u(x^+) + u(x^-))/2 = 0$; see the figure below. Since the graph (depicting the Fourier series) has discontinuities, the convergence can not be uniform.



Note that $u(x)$ is an odd function, so $a_k = 0$ for all k. For $k > 0$, again using the fact that u is odd,

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\sin kx\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\sin kx\,dx$$
$$= \frac{2}{\pi}\left(\left[-\frac{(\pi - x)\cos kx}{k}\right]_0^{\pi} - \frac{1}{k}\int_0^{\pi}\cos kx\,dx\right) = \frac{2}{\pi}\cdot\frac{\pi}{k} = \frac{2}{k}.$$

Hence

$$u(x) \sim 2\sum_{k=1}^{\infty}\frac{1}{k}\sin kx.$$

Letting $x = \pi/2$, we find that (with equality due to Dirichlet's theorem)

$$u(\pi/2) = 2\sum_{k=1}^{\infty}\frac{1}{k}\sin\frac{k\pi}{2} = 2\sum_{m=0}^{\infty}\frac{1}{2m+1}\sin\frac{(2m+1)\pi}{2} = 2\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1},$$

so since $u(\pi/2) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$, we obtain that

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1} = \frac{\pi}{4}.$$

Answer: $u(x) \sim 2\sum_{k=1}^{\infty}\frac{1}{k}\sin kx$; $\frac{\pi}{4}$.

4. We're looking for a solution to $y''(x) = -y'(x) + 9y(x+\pi) - 1 + 2\sin 3x$, so obviously y must be (at least) twice differentiable. Hence y' is continuous. This means that y'' must be continuous (since y solves the equation). Hence $y \in C^2$. Which means that $y'' \in C^2$, so $y \in C^4$ and so on. In other words, the solution must be very smooth.

- $y \in C^3$ implies that the Fourier series of y, y' and y'' converges to $y(x)$, $y'(x)$ and $y''(x)$, respectively (by Dirichlet's theorem). So, let $y(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$.
- y being 2π-periodical and $y''' \in E$ means we can form the termwise derivatives of y (with equality due to the first point):

$$y'(x) = \sum_{k=-\infty}^{\infty} ikc_k e^{ikx} \quad \text{and} \quad y''(x) = \sum_{k=-\infty}^{\infty} -k^2 c_k e^{ikx}.$$

Therefore, we can write the equation

$$y''(x) + y'(x) - 9y(x+\pi) = -1 + 2\sin 3x$$

as

$$\sum_{k=-\infty}^{\infty} (-k^2 + ik - 9e^{ik\pi})c_k e^{ikx} = -1 - ie^{i3x} + ie^{-i3x}$$

$$\Leftrightarrow \quad \sum_{k=-\infty}^{\infty} (-k^2 + ik - 9(-1)^k)c_k e^{ikx} = -1 - ie^{i3x} + ie^{-i3x}.$$

For y to be a solution to the differential equation, we must therefore (by uniqueness) have:

$$-k^2 + ik - 9(-1)^k = 0 \quad \text{or} \quad c_k = 0, \quad k \neq 0, \pm 3.$$

Obviously $c_k = 0$ is the only possibility when $k \neq 0, \pm 3$. If $k = 0$ we find that $-9c_0 = -1$, so $c_0 = 1/9$. If $k = \pm 3$, then

$$(-9 \pm 3i + 9)c_{\pm 3} = \mp i \quad \Leftrightarrow \quad c_{\pm 3} = -\frac{1}{3}.$$

Hence our solutions must have the form

$$y(x) = c_0 + c_{-3}e^{-i3x} + c_3 e^{i3x} = \frac{1}{9} - \frac{1}{3}\left(e^{-i3x} + e^{i3x}\right) = \frac{1}{9} - \frac{2}{3}\cos 3x.$$

Answer: $y(x) = \frac{1}{9} - \frac{2}{3}\cos 3x$.

5. The integral in the left-hand side is the one-sided convolution of u with $t \mapsto \cos 2t$, so taking the Laplace transform shows that

$$sU(s) - u(0) + U(s)\frac{s}{s^2+4} = \frac{5}{s}, \quad \operatorname{Re} s > 0.$$

Thus

$$U(s)\left(s + \frac{s}{s^2+4}\right) = 5 + \frac{5}{s} \quad \Leftrightarrow \quad U(s) \cdot \frac{s^3+5s}{s^2+4} = \frac{5+5s}{s}.$$

We solve for $U(s)$ and find that

$$U(s) = \frac{5(1+s)(s^2+4)}{s^2(s^2+5)} = \frac{5(s^3+s^2+4s+4)}{s^2(s^2+5)} = \frac{4}{s} + \frac{4}{s^2} + \frac{s+1}{s^2+5}.$$

By uniqueness, we obtain

$$u(t) = 4 + 4t + \cos\sqrt{5}t + \frac{1}{\sqrt{5}}\sin\sqrt{5}t.$$

Answer: $u(t) = u(t) = 4(1+t) + \cos\sqrt{5}t + \dfrac{1}{\sqrt{5}}\sin\sqrt{5}t.$

6. (a) We observe that $f \in G(\mathbf{R})$ so the Fourier transform exists.
   Easiest is to write the RHS as

$$f(t) = (H(t+\pi) - H(t-\pi))\sin t$$

and use $\mathcal{F}(u(t)\sin(t)) = \dfrac{1}{2i}(U(\omega+1) - U(\omega-1))$ with $u(t) = H(t+\pi) - H(t-\pi)$.
From the table, we find that $U(\omega) = \dfrac{2\sin\pi\omega}{\omega}$, so

$$\begin{aligned} F(\omega) &= \frac{1}{2i}\left(\frac{2\sin\pi(\omega-1)}{\omega-1} - \frac{2\sin\pi(\omega+1)}{\omega+1}\right) = \frac{1}{i}\left(-\frac{\sin\pi\omega}{\omega-1} + \frac{\sin\pi\omega}{\omega+1}\right) \\ &= i\sin\pi\omega\left(\frac{1}{\omega-1} - \frac{1}{\omega+1}\right) = \frac{2i\sin\pi\omega}{\omega^2-1} \end{aligned}$$

for $\mathrm{Re}\,\omega \neq \pm 1$ (extend by continuity). Alternatively, it is not an unreasonable approach to use direct integration:

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt = \int_{-\pi}^{\pi} \sin t\,e^{-i\omega t}\,dt = \frac{1}{2i}\int_{-\pi}^{\pi}\left(e^{it} - e^{-it}\right)e^{-i\omega t}\,dt \\ &= \frac{1}{2i}\left[\frac{e^{(1-\omega)it}}{i(1-\omega)} + \frac{e^{-i(1+\omega)t}}{-i(1+\omega)}\right]_{-\pi}^{\pi} \\ &= \frac{1}{2i}\left(e^{-i\omega\pi}\left(\frac{e^{i\pi}}{i(1-\omega)} + \frac{e^{-i\pi}}{i(1+\omega)}\right) - e^{i\omega\pi}\left(\frac{e^{-i\pi}}{i(1-\omega)} + \frac{e^{i\pi}}{i(1+\omega)}\right)\right) \\ &= \frac{1}{2}\left(e^{-i\omega\pi}\left(\frac{1}{1-\omega} + \frac{1}{1+\omega}\right) - e^{i\omega\pi}\left(\frac{1}{1-\omega} + \frac{1}{1+\omega}\right)\right) \\ &= -i\sin\pi\omega\left(\frac{1}{1-\omega} - \frac{1}{1-\omega}\right) = \frac{2i\sin\pi\omega}{\omega^2-1} \end{aligned}$$

with $\omega \neq \pm 1$. Direct calculation shows that $F(\pm 1) = \mp i\pi$.

(b) Notice that

$$\frac{\sin\pi\omega}{\omega^2-1} = \frac{1}{2i}\frac{2i\sin\pi\omega}{\omega^2-1} = \frac{1}{2i}F(\omega)$$

and since $D^{\pm}f(0)$ exists, Fourier inversion (Dirichlet's theorem for the Fourier transform) yields

$$\int_{-\infty}^{\infty}\frac{\sin\pi\omega}{\omega^2-1}\,d\omega = \frac{2\pi}{2i}\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R}F(\omega)\,e^{i\omega\cdot 0}\,d\omega = \frac{\pi}{i}\frac{f(0^+)+f(0^-)}{2} = 0.$$

Alternatively, we can use the fact that the integrand belongs to $L^1(\mathbf{R})$ and is an odd function, so the principal value must be zero.

For the second integral, observe that

$$\frac{\sin^2(\pi\omega)}{(\omega^2-1)^2} = \frac{1}{4}\left|\frac{2i\sin\pi\omega}{\omega^2-1}\right|^2 = \frac{1}{4}\left|F(\omega)\right|^2$$

and since $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, we can use Plancherel's theorem:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin^2\pi\omega}{(\omega^2-1)^2}\,d\omega &= \frac{2\pi}{4}\int_{-\infty}^{\infty} \left|f(x)\right|^2\,dx = \frac{\pi}{2}\int_{-\pi}^{\pi}\sin^2 t\,dt \\ &= \frac{\pi}{4}\int_{-\pi}^{\pi}(1-\cos 2t)\,dt = \frac{\pi^2}{2}.\end{aligned}$$

Answer: (a) $F(\omega) = \dfrac{2i\sin\pi\omega}{\omega^2-1}$, $\omega \neq \pm 1$, $F(\pm 1) = \mp\pi i$ (b) 0; $\pi^2/2$.

7. Let $u_n(x) = \dfrac{n^2+nx^3}{3n^2x^2+2nx+n^2}$. We find the pointwise limit as $n \to \infty$:

$$u_n(x) = \frac{1+x^3/n}{3x^2+2x/n+1} \to \frac{1}{3x^2+1}.$$

If the convergence is uniform, we can move the limit inside the integral. We prove that the convergence is uniform on $[0,1]$. We see that

$$\begin{aligned}\left|u_n(x) - \frac{1}{3x^2+1}\right| &= \left|\frac{\big(1+x^3/n\big)(3x^2+1) - \big(3x^2+2x/n+1\big)}{(3x^2+1)\big(3x^2+2x/n+1\big)}\right| \\ &= \left|\frac{3x^5/n + x^3/n - 2x/n}{(3x^2+1)\big(3x^2+2x/n+1\big)}\right| \leq \frac{3/n+1/n+2/n}{(0+1)\big(0+1\big)} = \frac{6}{n}.\end{aligned}$$

Clearly

$$\sup_{0\leq x\leq 1}\left|u_n(x) - \frac{1}{3x^2+1}\right| \leq \frac{6}{n} \to 0, \quad \text{as } n \to \infty.$$

Therefore the convergence is uniform on $[0,1]$ and

$$\lim_{n\to\infty}\int_0^1 u_n(x)\,dx = \int_0^1 \frac{1}{3x^2+1}\,dx = \left[\frac{1}{\sqrt{3}}\arctan\big(\sqrt{3}x\big)\right]_0^1 = \frac{\pi}{3\sqrt{3}}.$$

Answer: $\dfrac{\pi}{3\sqrt{3}}$.

Note: The bit about careful motivation is due to the problem of moving the limit inside the integral. We show that this is allowed by proving that the sequence is uniformly convergent on the interval we integrate over.