Transform theory 2026-01-07 – Solutions

1. (a) Yes. Since $u_n(x) \to 0$ for $|x| \le 1/2$ and $|\sin x^n - 0| \le |x|^n \le 2^{-n} \to 0$ independently of x, it is clear that the convergence is uniform.
   (b) Yes. The function belongs to $L^1(0,\infty)$ and is continuous, so the Laplace transform exists (at least) for $\operatorname{Re} s > 0$.
   (c) No. According to the Riemann-Lebesgue lemma, the Fourier coefficients for an L^1-function must tend to zero as $k \to \pm\infty$. This is clearly not the case here.
   (d) Yes. The partial sums are continuous functions and uniform convergence of the series therefore implies continuity.
   (e) Yes. This is clear since $3^{2k} = 9^k$ so we find this identity in the table (with $a = 9$).

   Answer: Yes. Yes. No. Yes. Yes.

2. (a) Assuming that $y, y' \in G$, we take the Fourier transform to find that

$$(i\omega)Y(\omega) + 2Y(\omega) = 5\,\mathcal{F}(e^{-3|x|}) = \frac{30}{9+\omega^2} \quad \Leftrightarrow \quad Y(\omega) = \frac{30}{(9+\omega^2)(2+i\omega)}.$$

   Hence, by decomposing into partial fractions,

$$Y(\omega) = \frac{1}{3-i\omega} + \frac{6}{2+i\omega} - \frac{5}{3+i\omega},$$

   where the substitution $s = i\omega$ can be helpful (but not necessary). From a table, we find that

$$\mathcal{F}(e^{3x}H(-x)) = \frac{1}{3-i\omega}, \quad \mathcal{F}(e^{-2x}H(x)) = \frac{1}{3+i\omega}, \quad \text{and} \quad \mathcal{F}(e^{2x}H(x)) = \frac{1}{2+i\omega},$$

   so by uniqueness and linearity,

$$y(x) = e^{3x}H(-x) + 6e^{-2x}H(x) - 5e^{-3x}H(x) = \begin{cases} e^{3x}, & x < 0, \\ 6e^{-2x} - 5e^{-3x}, & x > 0. \end{cases}$$

   We include either the value at 0^- or 0^+ (1 or $6 - 5 = 1$) to make the function continuous. This function and its piecewise derivative are absolutely integrable.
   (b) By the definition:

$$\mathcal{F}\left(e^{-x}H(x)\right) = \int_0^\infty e^{-x}e^{-i\omega x}\,dx = \int_0^\infty e^{-x(1+i\omega)}\,dx = \left[\frac{e^{-x(1+i\omega)}}{-(1+i\omega)}\right]_0^\infty$$
$$= 0 - \frac{e^{-0(1+i\omega)}}{-(1+i\omega)} = \frac{1}{1+i\omega}.$$

   Answer: $y(x) = \begin{cases} e^{3x}, & x < 0, \\ 6e^{-2x} - 5e^{-3x}, & x \ge 0. \end{cases}$; see above.

3. We assume that y, y', y'' all belong to X_a (and verify this at the end). Taking the Laplace transform of the equation, we obtain that

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) - 4\big(sY(s) - y(0)\big) + 5Y(s) &= \frac{4}{s-1} \\ \Leftrightarrow \quad Y(s)\left(s^2 - 4s + 5\right) &= s - 1 + \frac{4}{s-1}, \quad \operatorname{Re} s > 1. \end{aligned}$$

Hence, since $s^2 - 4s + 5 = (s-2)^2 + 1$,

$$\begin{aligned} Y(s) &= \frac{s-1}{(s-2)^2+1} + \frac{4}{(s-1)((s-2)^2+1)} = \frac{s-1}{(s-2)^2+1} + \frac{2}{s-1} + \frac{6-2s}{(s-2)^2+1} \\ &= -\frac{s-5}{(s-2)^2+1} + \frac{2}{s-1} = -\frac{s-2}{(s-2)^2+1} + \frac{3}{(s-2)^2+1} + \frac{2}{s-1} \end{aligned}$$

after decomposing into partial fractions and rewriting. Using the rule $\mathcal{L}(e^{2t}u(t)) = U(s-2)$ and

$$\mathcal{L}(e^t) = \frac{1}{s-1}, \quad \mathcal{L}(\cos t) = \frac{s}{s^2+1}, \quad \text{and} \quad \mathcal{L}(\sin t) = \frac{1}{s^2+1},$$

for $\operatorname{Re} s > 1$. Hence

$$y(t) = e^{-4t} + te^{-t} + e^{2t}$$

by uniqueness and linearity. Obviously y and its derivatives are exponentially bounded.

Answer: $y(t) = 2e^t + e^{2t}\,(3\sin t - \cos t)$, $t \geq 0$.

4. The sum in the equation is the convolution of u with the function $k \mapsto 2k+1$ (for $k \geq 0$). We take the Z transform of the equation and find that

$$U(z)\left(\frac{2z}{(z-1)^2} + \frac{z}{z-1}\right) = \frac{6z}{z-2},$$

where we assume that (at least) $|z| > 2$. Reformulating this equation, we find that

$$U(z) = \frac{6z}{z-2}\left(\frac{2z + z(z-1)}{(z-1)^2}\right)^{-1} = z\,\frac{6(z-1)^2}{z(z+1)(z-2)}.$$

Decomposing into partial fractions yields

$$z\left(-\frac{3}{z} + \frac{8}{z+1} + \frac{1}{z-2}\right) = -3 + \frac{8z}{z+1} + \frac{z}{z-2}.$$

From a table, we find that

$$\mathcal{Z}\left(\delta[n]\right) = 1, \quad \mathcal{Z}\left((-1)^n\right) = \frac{z}{z+1} \quad \text{and} \quad \mathcal{Z}\left(2^n\right) = \frac{z}{z-2},$$

so we obtain by linearity and uniqueness that

$$u[n] = -3\delta[n] + 8(-1)^n + 2^n.$$

Answer: $u[n] = -3\delta[n] + 8(-1)^n + 2^n$, $n = 0, 1, 2, 3, \ldots$.

5. From the table, we find that if $U(\omega) = \dfrac{2\sin\omega}{\omega}$, then $u(x) = 1$ for $-1 \le x \le 1$ and $u(x) = 0$ elsewhere has this Fourier transform. Fourier inversion (Dirichlet's theorem for the Fourier transform) yields

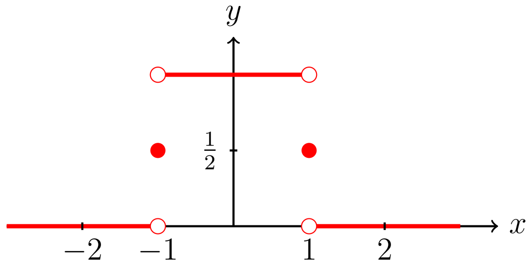
$$g(x) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \frac{2\sin\omega}{\omega} e^{i\omega\cdot x}\, d\omega = \frac{u(x^+) + u(x^-)}{2}$$

since $u(x)$ is piecewise continuous (even constant) and has right- and left-hand derivatives at all points.

We see that the expression above gives

$$g(x) = \begin{cases} 1, & |x| < 1, \\ 1/2, & x = \pm 1, \\ 0, & |x| > 1, \end{cases}$$

which can be seen graphically to the right.



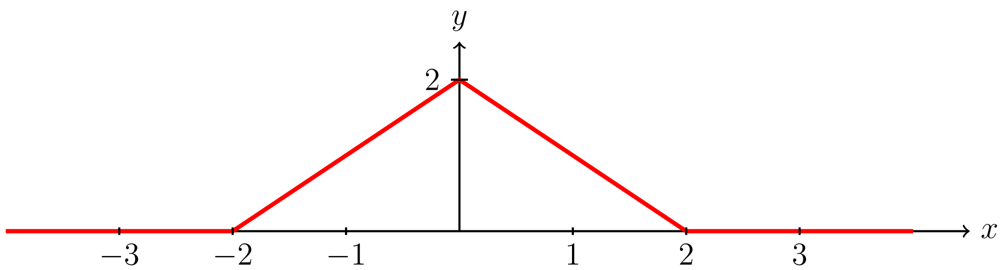
The convolution $g * g$ has the Fourier transform

$$\mathcal{F}(g*g) = \left(\frac{2\sin\omega}{\omega}\right)^2 = \frac{4\sin^2\omega}{\omega^2} = \frac{2(1-\cos 2\omega)}{\omega^2},$$

which we from the table is the transform of $(2 - |x|)(H(x+2) - H(x-2))$. Therefore, by uniqueness,

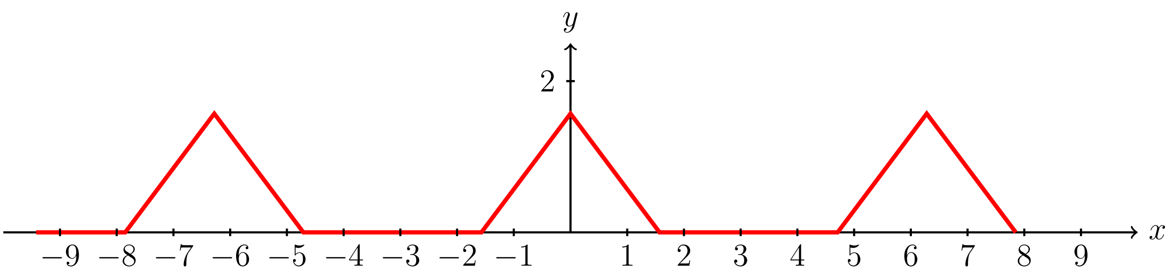
$$g * g(x) = \mathcal{F}^{-1}\left(\frac{2(1-\cos 2\omega)}{\omega^2}\right) = (2-|x|)(H(x+2) - H(x-2))$$

if $g * g$ is continuous, which is true since at least one factor (well..) is bounded.



Answer: $g(x) = 1$ when $|x| < 1$, $g(\pm 1) = 1/2$ and $g(x) = 0$ when $|x| > 1$; see above.

6. Clearly $u \in E$. This is obvious since the the periodic extension function is continuous for all x.



Recall that this also implies that $u \in L^2(-\pi,\pi)$.

We note that $u(x)$ is an even function, so $b_k = 0$ for all k. For $k > 0$, again using the fact that u is even,

$$
\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\cos kx\,dx = \frac{2}{\pi}\int_0^{\pi/2} (\pi - x)\cos kx\,dx \\
&= \frac{2}{\pi}\left(\left[\frac{(\pi - x)\sin kx}{k}\right]_0^{\pi/2} + \frac{1}{k}\int_0^{\pi/2} \sin kx\,dx\right) = \frac{2}{\pi k}\left[-\frac{\cos kx}{k}\right]_0^{\pi/2} \\
&= \frac{2}{\pi k^2}\left(1 - \cos\frac{k\pi}{2}\right)
\end{aligned}
$$

and

$$
a_0 = \frac{2}{\pi}\int_0^{\pi/2} (\pi - x)\,dx = \frac{2}{\pi}\left[-\frac{(x - \pi/2)^2}{2}\right]_0^{\pi/2} = \frac{\pi}{4}.
$$

Hence

$$
u(x) \sim \frac{\pi}{4} + \sum_{k=1}^{\infty} \frac{2}{\pi k^2}\left(1 - \cos\frac{k\pi}{2}\right)\cos kx.
$$

Next we use Parseval's identity:

$$
\begin{aligned}
\frac{|a_0|^2}{2} + \sum_{k=1}^{\infty}\left(|a_k|^2 + |b_k|^2\right) &= \frac{|a_0|^2}{2} + \sum_{k=1}^{\infty}|a_k|^2 \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} |(\pi/2 - |x|)|^2\,dx = \frac{2}{\pi}\int_0^{\pi/2} (\pi/2 - x)^2\,dx \\
&= \frac{2}{\pi}\left[-\frac{(\pi/2 - x)^3}{3}\right]_0^{\pi/2} = \frac{\pi^2}{12}.
\end{aligned}
$$

So using this equality, we see that

$$
\frac{\pi^2}{12} = \frac{(\pi/4)^2}{2} + \sum_{k=1}^{\infty}\frac{4}{\pi^2 k^4}\left(1 - \cos\frac{k\pi}{2}\right)^2 \quad\Rightarrow\quad \frac{\pi^2}{12} - \frac{(\pi/4)^2}{2} = \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^4}\left(1 - \cos\frac{k\pi}{2}\right)^2
$$

so

$$
\sum_{k=1}^{\infty}\frac{1}{k^4}\left(1 - \cos\frac{k\pi}{2}\right)^2 = \frac{\pi^2}{4}\left(\frac{\pi^2}{12} - \frac{(\pi/4)^2}{2}\right) = \frac{5\pi^4}{384}.
$$

Answer: $u(x) \sim \dfrac{\pi}{4} + \displaystyle\sum_{k=1}^{\infty}\frac{2}{\pi k^2}\left(1 - \cos\frac{k\pi}{2}\right)\cos kx;\ \dfrac{5\pi^4}{384}.$

7. Let $u_k(x) = \dfrac{\sin k^2 x}{x^2 + k^4}$. Clearly

$$
|u_k(x)| \le \frac{1}{k^4}, \quad k = 1, 2, 3, \ldots
$$

so the series defining $u(x)$ is convergent for all x (actually uniformly convergent by the M-test). To show that $u(x)$ is differentiable, we prove that

$$
\sum_{k=1}^{\infty} u_k'(x) = \sum_{k=1}^{\infty}\left(\frac{k^2\cos k^2 x}{x^2 + k^4} + \frac{-2x\sin k^2 x}{(x^2 + k^4)^2}\right)
$$

is uniformly convergent. We see that

$$\left|\frac{k^2\cos k^2x}{x^2+k^4}+\frac{-2x\sin k^2x}{(x^2+k^4)^2}\right|\le\frac{k^2}{x^2+k^4}+\frac{2|x|}{(x^2+k^4)^2}\le\frac{1}{k^2}+\frac{2|x|}{(x^2+k^4)^2}.$$

Furthermore,

$$|x|\le k \quad\Rightarrow\quad \frac{2|x|}{(x^2+k^4)^2}\le\frac{2k}{k^8}=\frac{2}{k^7}$$

and

$$|x|>k \quad\Rightarrow\quad \frac{2|x|}{(x^2+k^4)^2}\le\frac{2|x|}{|x|^4}=\frac{2}{|x|^3}\le\frac{2}{k^3},$$

so it is clear that $v(x)=\sum_{k=1}^{\infty}u_k'(x)$ is uniformly convergent by the M-test. Since all u_k' are continuous, it follows that v is continuous. The fact that $u(x)=\sum_{k=1}^{\infty}u_k(x)$ is convergent and that the series defining the continuous function v is uniformly convergent, we obtain that $u'(x)=v(x)$, which proves that $u\in C^1(\mathbf{R})$.

Answer: See above.