Lecture 1: Introduction, Periodic Functions and Series

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"It's Showtime!"
—Ben Richards

1 Preliminaries

The prerequisites for this course is basically single variable analysis, multivariate analysis and linear algebra. Some complex analysis is helpful but I'll make the course self-contained with respect to that.

1.1 Complex-valued Functions

We will immediately start working with complex valued functions of a real variable (at this point, we'll consider complex valued functions of a complex variable later on). If you've taken a course in complex analysis, everything will be familiar. If not, we do not need too much complex analysis (although complex numbers will be everywhere). Let's make a couple of general definitions for the things that we will need.

Definition. We write that $\lim_{z\to z_0} f(z) = A$ for some $A \in \mathbf{C}$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|z - z_0| < \delta \quad \Rightarrow \quad |f(z) - f(z_0)| < \epsilon.$$

We call f continuous at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$.

So the definition is almost identical with the real case, it's just that $|\cdot|$ is now the complex absolute value (meaning that $|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$). Similarly to the real case, continuity can equivalently be phrased in terms of sequences (Heine's definition): for any sequence $z_n \to z_0$ we have $f(z_n) \to f(z_0)$. This description is sometimes easier to deal with than Cauchy's δ-ϵ-definition.

At this point, we will mainly consider functions $u\colon \mathbf{R} \to \mathbf{C}$. For functions of this type, we can always write $u(x) = \alpha(x) + i\beta(x)$, where $\alpha, \beta\colon \mathbf{R} \to \mathbf{R}$ are real-valued functions (the real and imaginary part of $u(x)$). Operations like differentiation and integration works like expected. We treat the real and imaginary part separately and then sum the results, i.e.,

$$u'(x) = \alpha'(x) + i\beta'(x) \quad \text{and} \quad \int_a^b u(x)\,dx = \int_a^b \alpha(x)\,dx + i\int_a^b \beta(x)\,dx.$$

This simplifies matters. In the case when we need to consider functions of a complex variable, things get a bit trickier, but that can wait until the second half of the course. This decomposition into real- and imaginary parts of the function $u(x)$ is sufficient for what we need right now.

2 Periodic Functions

A function $u\colon \mathbf{R} \to \mathbf{C}$ is called **periodic** if there is some constant $T > 0$ such that

$$u(x+T) = u(x) \text{ for every } x \in \mathbf{R}.$$

Note that if u is T-periodic, then u is also $2T$-periodic since

$$u(x+2T) = u(x+T+T) = u(x+T) = u(x) \text{ for every } x \in \mathbf{R}.$$

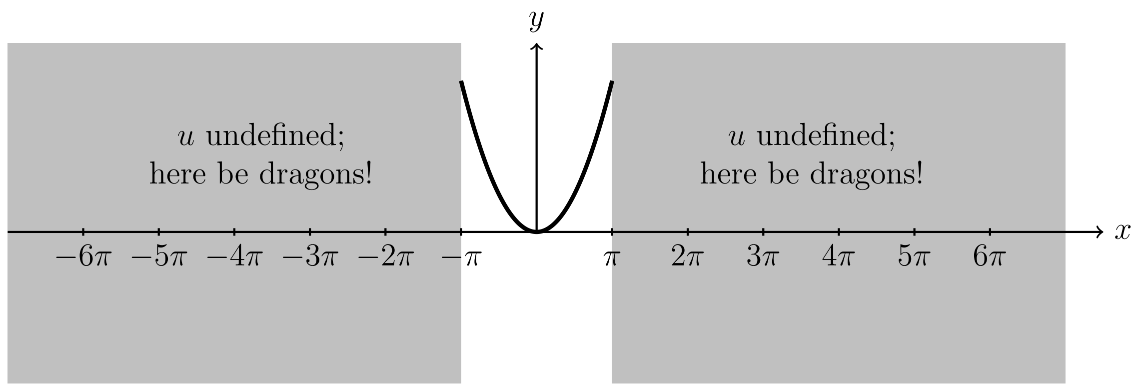
And similarly, u is nT periodic for $n = 1, 2, 3, \ldots$. We usually refer to the smallest possible period T when referring to a function's period. A constant function does not have a smallest period (but is obviously periodic).

Example

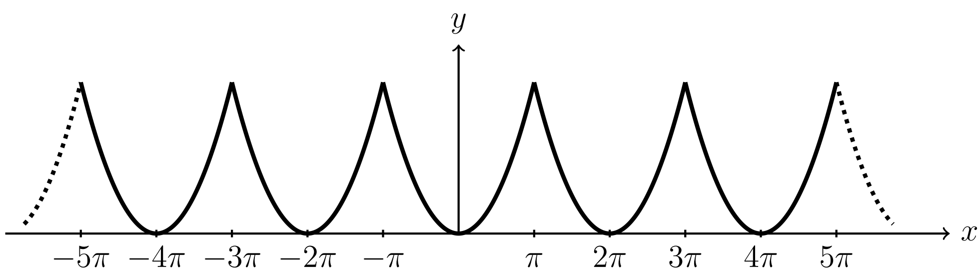
(i) The functions $\sin t$ and $\cos t$ are 2π-periodic functions.

(ii) The functions e^{int} are $\dfrac{2\pi}{n}$-periodic functions.
These functions are usually known as *harmonic oscillations*.

In this course, we will mainly be considering 2π-periodic functions. How would we handle a function that is not periodic? Consider a function $u\colon [-\pi, \pi] \to \mathbf{C}$. This means that u is undefined outside the interval $[-\pi, \pi]$. For example, the graph (for a real example) could look something like below.



From a function $u\colon [a, b[\to \mathbf{C}$ defined on an interval $[a, b[$ (say $[-\pi, \pi[$), we can consider the periodic extension of u that is defined for all $x \in \mathbf{R}$ such that $u(x+T) = u(x)$ for every x, where $T = b - a$. Similarly for $]a, b]$. For the function above, the periodic extension would look like the graph below.



Integrating periodic functions

If u is an integrable periodic function with period T, note that $\int_0^T u(x)\,dx = \int_a^{a+T} u(x)\,dx$ for any $a \in \mathbf{R}$. Therefore we can choose any integration domain of length T and to make the notation more compact, we sometimes write $\int_T u(x)\,dx$ to indicate that we integrate over one period of the function.

3 Function Spaces

Let's start with defining two rather general spaces.

$L^1(a, b)$

Definition. We define the space $L^1(a, b)$ to consist of those functions $u\colon]a, b[\to \mathbf{C}$ for which

$$\int_a^b |u(x)|\,dx < \infty.$$

In other words, we collect those functions that are absolutely integrable on $[a, b]$.

$L^2(a, b)$

Definition. We define the space $L^2(a, b)$ to consist of those functions $u\colon]a, b[\to \mathbf{C}$ for which

$$\int_a^b |u(x)|^2\,dx < \infty.$$

These definitions might look fairly innocuous, but there's some stuff buried here. First and foremost, we really should be using a different type of integral in the place of the Riemann integral that we're used to (the Lebesgue counterpart is more suitable). However, in the case where the function is Riemann integrable, these two integrals coincide so we can live with this problem in this course. There's more issues hiding around the corner, and we'll get to some of these next lecture. The way we will handle this in this course is to restrict our attention to a subset of $L^2(a, b)$ where these problems are nonexistent.

Piecewise continuous function

Definition. We call a function u on an interval $[a, b]$ piecewise continuous if there are a finite number of points such that u is continuous everywhere on $[a, b]$ except for at these points. Moreover, if $c \in]a, b[$ is one of these points, the limits

$$\lim_{x\to c^-} u(x) \quad \text{and} \quad \lim_{x\to c^+} u(x)$$

exist. We denote the space of all piecewise continuous functions on an interval $[a, b]$ by $E[a, b]$, or just E if the interval is clear from the context.

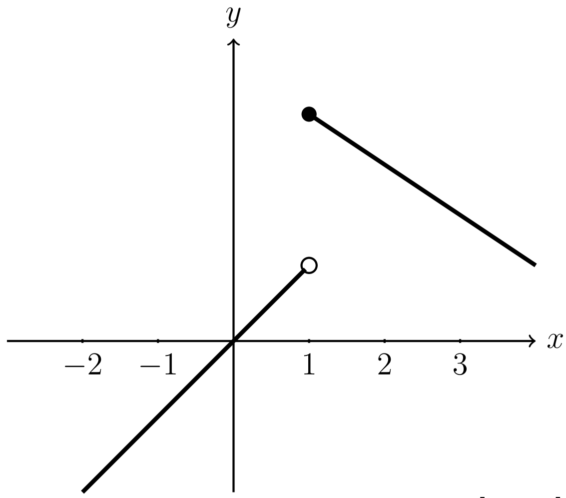
We will denote the left- and righthand limits at a point c by

$$u(c^-) = \lim_{x \to c^-} u(x) \quad \text{and} \quad u(c^+) = \lim_{x \to c^+} u(x),$$

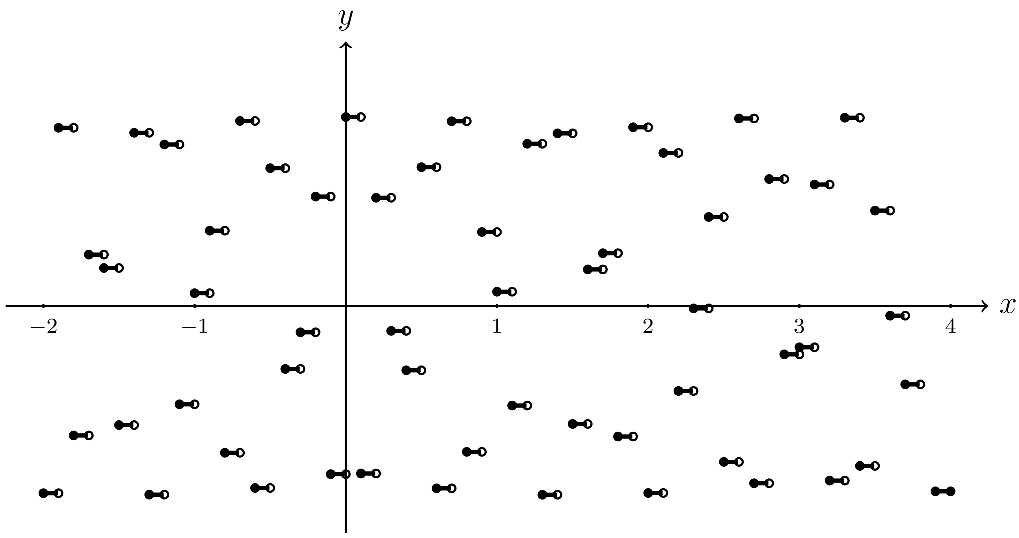
respectively.

As an example, we could consider the function

$$f(x) = \begin{cases} x, & -2 \leq x < 1, \\ 4 - x, & 1 \leq x \leq 3. \end{cases}$$



We might consider something more dramatic as well. The function below is in $E[-2, 4]$ (it is in fact even piecewise constant).



So you probably get the point. We can cover quite a large amount of function types by only considering piecewise continuous functions. However, this thinking might be a bit disingenuous. It should be noted that this class of functions is still extremely small compared to, say, $L^2(2, 4)$.

3.1 Left- and Righthand Derivatives

For $u \in E$, we define the left- and righthand derivatives at a point $x \in]a, b[$ by

$$D^- u(x) = \lim_{h \to 0^-} \frac{u(x+h) - u(x^-)}{h} \quad \text{and} \quad D^+ u(x) = \lim_{h \to 0^+} \frac{u(x+h) - u(x^+)}{h}$$

if the limit exist. For the endpoints, we only define $D^+ u(a)$ and $D^- u(b)$, respectively.

The space $E'[a,b]$

Definition. The linear space $E'[a,b]$ consists of those $u \in E[a,b]$ such that $D^- u(x)$ exists for $a < x \leq b$ and that $D^+ u(x)$ exists for $a \leq x < b$.

Note the following.

Properties

(i) If u is continuous, then $u \in E$.

(ii) If u is differentiable, then $u \in E'$.

(iii) On a compact interval, $E' \subset E \subset L^2 \subset L^1$ (that $L^2 \subset L^1$ follows from Cauchy-Schwarz).

4 Series

As we remember from TATA42, we define a numerical series S of a sequence $a_0, a_1, a_2, \ldots$ by

$$S = \sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k$$

whenever this limit exists (this is the *definition* of a convergent series). We have also studied certain types of *functional series*:

$$S(x) = \sum_{k=0}^{\infty} u_k(x) = \lim_{n\to\infty} \sum_{k=0}^{n} u_k(x)$$

for those x where the limit exists. In particular, we've seen power series where $u_k(x) = c_k x^k$ and c_k are real (or complex) constants. The sums

$$S_n(x) = \sum_{k=0}^{n} u_k(x), \quad n \in \mathbf{N},$$

are called the **partial sums** of the series S. Whenever $S_n(x)$ has a limit as $n \to \infty$, this is the value of $S(x)$. We call the limit $S(x)$ the **pointwise** limit of $S_n(x)$ as $n \to \infty$. In other words, the partial sums $S_n(x)$ converges **pointwise** to $S(x)$. There are other types of convergence as we shall see later on.

5 Fourier Series

Let $u \in L^1(-\pi, \pi)$ and define

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) \cos kx \, dx \quad \text{and} \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) \sin kx \, dx.$$

The series

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \big(a_k \cos kx + b_k \sin kx\big)$$

is called the **real Fourier series** of the function u. The real constants a_k and b_k (if u is real) are called the **Fourier coefficients** of u.

We will write that

$$u(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) .$$

Why not equality? Well, there's a couple of problems here.

(i) For a given $x \in [-\pi, \pi]$, does $S(x)$ exist? That is, does the series converge?

(ii) If $S(x)$ does exist, is it true that $S(x) = u(x)$?

(iii) If we consider $u \in L^1(-\pi, \pi)$, what does $u(x)$ even mean?

(iv) Suppose that $S(x)$ does exist and that $S(x) = u(x)$, in what way do we expect the partial sums to converge?

So when we write that $u(x) \sim S(x)$ we mean that $S(x)$ is the expression that we obtain from u when calculating the Fourier series. Over the next couple of lectures we will answer the questions above.

Example

Suppose that $u(x) = \operatorname{sgn}(x)$ for $x \in [-\pi, \pi[$, where $\operatorname{sgn}(x) = -1$ when $x < 0$, $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(x) = 1$ when $x > 0$. Find the Fourier series of u.

Solution. We consider the periodic extension of u. The Fourier coefficients can be calculated as follows:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) \cos(0 \cdot x)\, dx = \frac{1}{\pi}(-1+1) = 0,$$

and for $k \geq 1$,

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) \cos kx\, dx = \frac{1}{\pi} \left(\int_{-\pi}^{0} -\cos kx\, dx + \int_{0}^{\pi} \cos kx\, dx \right) \\ &= \frac{1}{\pi} \left(\left[-\frac{\sin kx}{k} \right]_{-\pi}^{0} + \left[\frac{\sin kx}{k} \right]_{0}^{\pi} \right) \\ &= \frac{1}{\pi} \left(-\frac{-\sin(-k\pi)}{k} + \frac{\sin k\pi}{k} \right) = \frac{2 \sin k\pi}{k} = 0, \end{aligned}$$

and finally for $k \geq 1$

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) \sin kx\, dx = \frac{1}{\pi} \left(\int_{-\pi}^{0} -\sin kx\, dx + \int_{0}^{\pi} \sin kx\, dx \right) \\ &= \frac{1}{\pi} \left(\left[\frac{\cos kx}{k} \right]_{-\pi}^{0} + \left[\frac{-\cos kx}{k} \right]_{0}^{\pi} \right) \\ &= \frac{1}{\pi} \left(\frac{1}{k} - \frac{\cos k\pi}{k} + \frac{-\cos k\pi}{k} + \frac{1}{k} \right) = \frac{2 - 2\cos k\pi}{k\pi} = \frac{2(1-(-1)^k)}{k\pi}. \end{aligned}$$

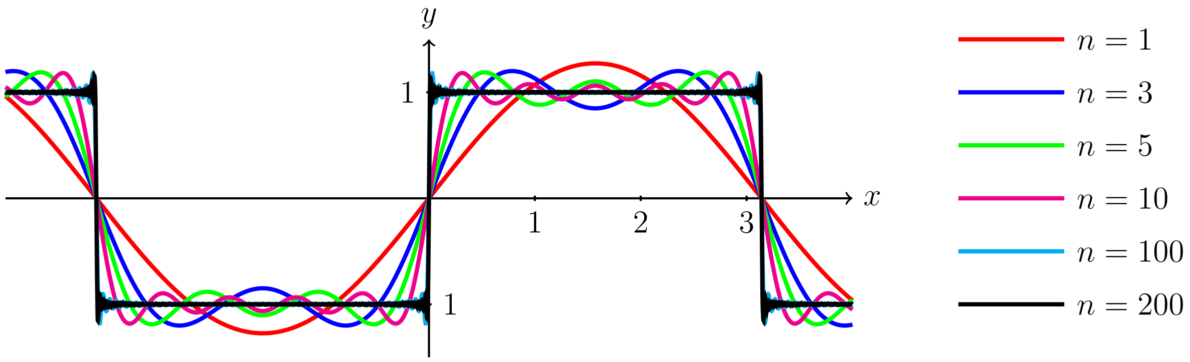
Hence

$$u(x) \sim \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1-(-1)^k}{k} \cdot \sin kx.$$

Now, a reasonable question is: "does this series converge?" Since, if k is odd,

$$\left| \frac{1-(-1)^k}{k} \cdot \sin kx \right| = \frac{2}{k} |\sin kx|,$$

it is *not* absolutely convergent (for most x). The series passes the divergence test, but that only means we cannot conclude that it is divergent. It might be tempting to think of Leibniz, but this series is not alternating (we might find some values for x but not in general). So we don't know if the series converges or diverges for just about any value of x. Don't worry, we'll get to this. In fact, this series is actually convergent to $u(x)$ for just about every x, but we have no idea why at this point. Summing the first n terms, we find the graphs below. This indicates that the sum indeed converges to the desired function, but there's some "squiggly" stuff going on around the jump points. We'll get back to that as well.



5.1 Complex Fourier series

So when examining the example in the previous section, we see that the same type of calculations are repeated for cos and sin. Considering that we've seen this phenomenon previously in analysis courses, might we consider a complex form instead and obtain both results at once? The answer is yes.

Similarly to above, let $u \in L^1(-\pi, \pi)$ and define

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) e^{-ikx} \, dx.$$

The series

$$u(x) \sim S(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

is called the **complex Fourier series** of u and c_k are the **complex Fourier coefficients** of u. In this case, we define the partial sums $S_n(x) = \sum_{k=-n}^{n} u_k(x)$ so that we sum symmetrically around $k = 0$. Note that this gives a different type of convergence than if we were to have two different limits.

So how does this connect to the real Fourier series? Well, if we recall Euler's formulas, we have

$$e^{ikx} = \cos kx + i \sin kx.$$

Thus we see that

$$c_{\pm k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\big(\cos(\pm kx) - i\sin(\pm kx)\big)\,dx = \frac{1}{2}\left(a_k \mp ib_k\right),$$

and therefore, for $k > 0$,

$$\begin{aligned} c_k e^{ikx} + c_{-k}e^{-ikx} &= \frac{1}{2}\left(a_k - ib_k\right)\left(\cos kx + i\sin kx\right) + \frac{1}{2}\left(a_k + ib_k\right)\left(\cos kx - i\sin kx\right) \\ &= \frac{1}{2}\left(2a_k\cos kx + 2b_k\sin kx\right) = a_k\cos kx + b_k\sin kx. \end{aligned} \tag{1}$$

Hence the two types of partial sums (the real and the complex) are equal, so they converge to the same thing if convergent (which they are at the same time). The condition that $u \in L^1(-\pi,\pi)$ is natural in the sense that this will ensure that the Fourier coefficients exist as absolutely convergent integrals:

$$|c_k| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx\right| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|u(x)|\left|e^{-ikx}\right|\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}|u(x)|\,dx.$$

When dealing with the complex Fourier coefficients, there are several different notations that are quite common. We might use these at certain points:

$$c_k = \widehat{u}_k = \widehat{u}[k], \quad k \in \mathbf{Z}.$$

So which representation is the best? That depends on the situation. The real series is clearly real valued (if u is real valued), which might be nice to see when working with real functions. However, the complex series is more compact and you can do more calculations at the same time. So the choice is basically yours, but be aware that you need to be able to handle both variants to pass the course. There's also some slight differences in function spaces used, so be careful which series you work with. In these notes, most things will be carried out using the complex form, whereas the book does most things with the real form. So there. You can choose yourself.

6 Does the Fourier Series Care About the Pointwise Function?

One thing to note in particular is the fact that the Fourier coefficients are defined by integrals of the function we're working with, say u, multiplied by sinusoids. This implies (among other things) that the Fourier coefficients don't care about what happens at *every* point. We could redefine $u(x)$ at points (certainly finitely many but it could be a lot worse) and the Fourier coefficients would turn out to be the same. What does this mean?

- Without prior knowledge about the function u (like continuity etc), there's no way to recover u from the Fourier coefficients *at every point*.

- When considering elements from $L^p(\mathbf{R})$-spaces, one typically considers these as *equivalence classes* where functions differing on a small set are considered the same (small meaning measure zero). We will *not* do this, but consider functions defined at every point.

7 Frequency Domain?

Another thing that's straightforward with the complex notation is that we can plot some graphs that describe the "frequency content" of a periodic function. Consider the function

$$u(x) = 1 + 3\cos x - 2\cos 2x + 6\cos 4x + 4\cos 7x.$$

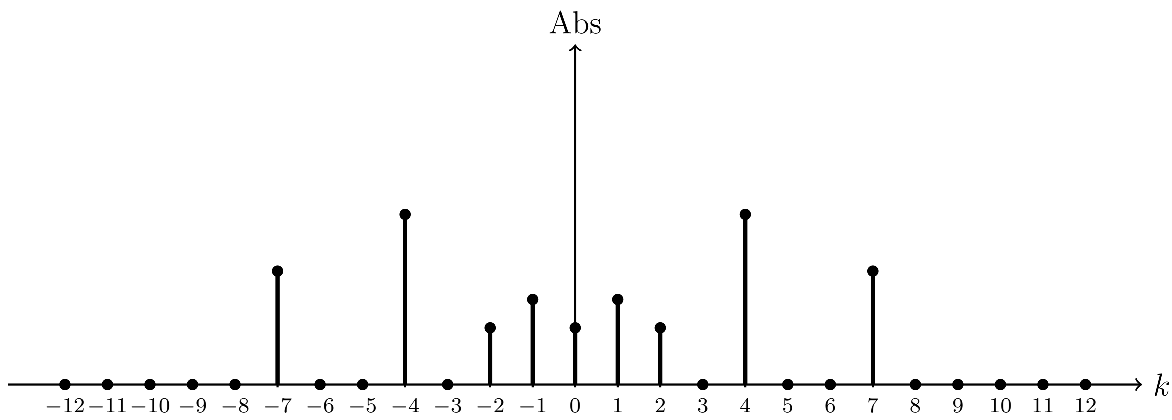
Using Euler's formulas, we can rewrite this as

$$u(x) = 1 + \frac{3}{2}e^{ix} + \frac{3}{2}e^{-ix} - e^{i2x} - e^{-i2x} + 3e^{i4x} + 3e^{-i4x} + 2e^{i7x} + 2e^{-i7x}.$$

This is the Fourier series for $u(x)$, although this is not exactly clear at the moment since we haven't shown any results regarding the uniqueness. As an exercise, try to use this representation to calculate the Fourier coefficients. You'll find that

$$c_0 = 1, \quad c_{\pm 1} = \frac{3}{2}, \quad c_{\pm 2} = -1, \quad c_{\pm 4} = 3, \quad \text{and } c_{\pm 7} = 2.$$

All other $c_k = 0$. What we usually do is draw the magnitude $|c_k|$ of the coefficients c_k (remember that they might be complex as well as negative). For this example, this would look like the graph below.



From this graph we see what frequencies are needed to represent a periodic function. This type of plot will become more interesting when we consider the Fourier transform instead. If we have used a real Fourier series, the magnitude is given by $\sqrt{a^2 + b^2}$ and we only plot for nonnegative k (why?); see equation (1) for the connection.

For something a little messier, let's consider the following.

Example

Let $u(x) = \cos\left(\frac{x}{2}\right)$, $-\pi \leq x \leq \pi$, and find the Fourier series of u. Draw a magnitude plot.

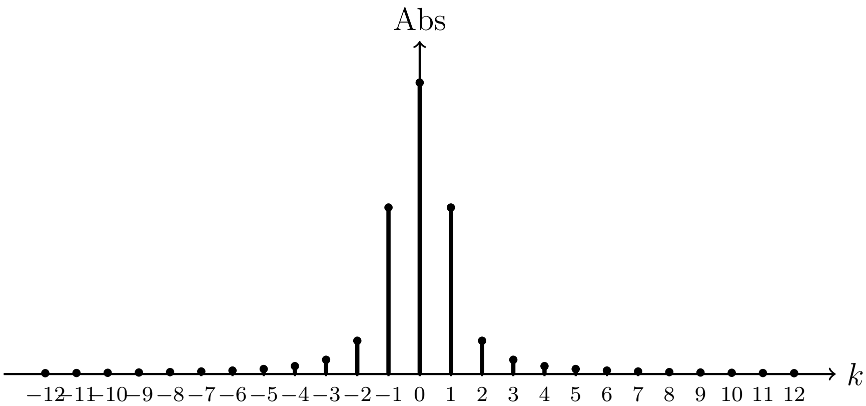
Solution. We need the Fourier coefficients, so

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\left(\frac{x}{2}\right)\, dx = \frac{2}{\pi}$$

and for $k \neq 0$,

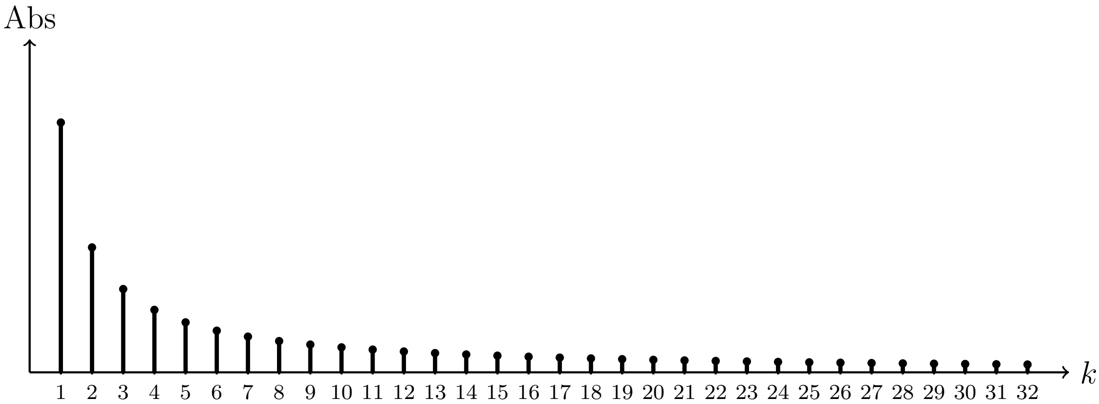
$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}\cos\left(\frac{x}{2}\right)\, dx = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(e^{-ikx+ix/2} + e^{-ikx-ix/2}\right)\, dx \\ &= \frac{1}{4\pi}\left[\frac{e^{-ikx+ix/2}}{i(-k+1/2)} + \frac{e^{-ikx-ix/2}}{i(-k-1/2)}\right]_{-\pi}^{\pi} = \frac{(-1)^k}{2\pi}\left(\frac{1}{-k+1/2} - \frac{1}{-k-1/2}\right) \\ &= \frac{(-1)^{k+1}}{2\pi}\left(\frac{1}{(-k+1/2)(-k-1/2)}\right) = \frac{4(-1)^{k+1}}{2\pi(4k^2-1)}. \end{aligned}$$

We note that $c_k = \dfrac{4(-1)^{k+1}}{2\pi(4k^2-1)}$ for all $k \in \mathbf{Z}$, so $u(x) \sim \displaystyle\sum_{k=-\infty}^{\infty} \frac{4(-1)^{k+1}}{2\pi(4k^2-1)}\, e^{ikx}$.



We note that $c_k \neq 0$ for every $k \in \mathbf{Z}$ (they do tend to zero quite fast however), unlike the previous example where only certain values of k where nonzero. If only a finite number of c_k are nonzero, this means that the function is a trigonometric polynomial that is periodic with period 2π. While $\cos(x/2)$ is periodic, it is not periodic with period 2π. This is an important distinction.

So how does the spectrum of $\operatorname{sgn}(x)$, $-\pi \leq x \leq \pi$ look? Recall that for odd k, we showed previously that $b_k = 4/k\pi$ and for even k, $b_k = 0$ (all $a_k = 0$). Thus we obtain the plot below.



8 Even/Odd Functions

Recall that a function u is even if $u(-x) = u(x)$ and odd if $u(-x) = -u(x)$. The most common examples being that $u(x) = \cos x$ is even and $u(x) = \sin x$ is odd. For functions that espouse these additional symmetries, we can make some simplifications to the Fourier calculations.

Theorem.

(i) If u is even, then $b_k = 0$ for $k = 1, 2, 3, \ldots$.

(ii) If u is odd, then $a_k = 0$ for $k = 1, 2, 3, \ldots$.

Proof. If u is even, then the product $u(x)\sin kx$ is odd for $k = 1, 2, 3, \ldots$. Hence

$$\int_{-\pi}^{\pi} u(x)\sin kx\,dx = 0,$$

so $b_k = 0$. Similarly, if u is odd, then $u(x)\cos kx$ is odd for $k = 1, 2, 3, \ldots$ which implies that $a_k = 0$. □

Example

Find the Fourier series for $u(x) = x^2$, $x \in [-\pi, \pi]$.

Solution. First alternative: the real form. Since u is even, we know that $b_k = 0$. This means that we'll obtain a pure cosine-series. With this in mind, we calculate

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^2}{3}$$

and

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos kx\,dx = /\ x^2\cos kx \text{ even}\ / = \frac{2}{\pi}\int_0^{\pi} x^2\cos kx\,dx \\ &= /\ \text{I.B.P.}\ / = \frac{2}{\pi}\left(\left[\frac{x^2\sin kx}{k} + \frac{2x\cos kx}{k^2}\right]_0^{\pi} - \frac{2}{k^2}\int_0^{\pi}\cos kx\,dx\right) \\ &= \frac{2}{\pi}\left(\frac{2\pi\cos(\pi k)}{k^2}\right) = \frac{4(-1)^k}{k^2}. \end{aligned}$$

Alternative two: the complex form. Ignoring for a moment that we know that u is even, we could just do the calculation for the complex Fourier coefficients without using any additional information. Indeed,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}$$

and for $k \neq 0$:

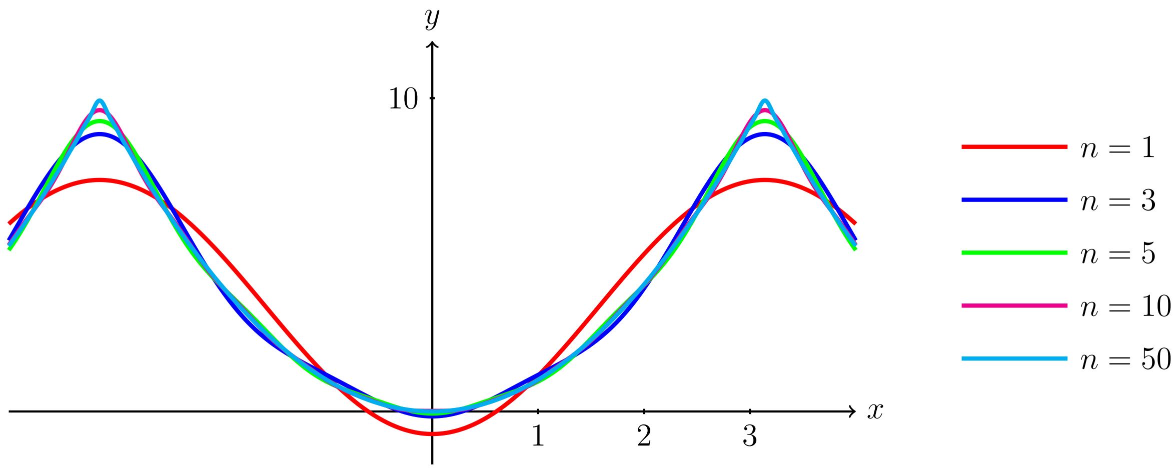
$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-ikx}\,dx = /\ \text{I.B.P.}\ / = \frac{1}{2\pi}\left(\left[-\frac{1}{ik}x^2 e^{-ikx} + \frac{2x}{k^2}e^{-ikx}\right]_{-\pi}^{\pi} - \frac{2}{k^2}\int_{-\pi}^{\pi} e^{-ikx}\,dx\right) \\ &= \frac{1}{2\pi}\left(\frac{4\pi(-1)^k}{k^2}\right) \end{aligned}$$

Due to the symmetry $c_{-k} = c_k$, we obtain the same pure cosine series as before.

So we have shown that

$$u(x) \sim \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos kx.$$

We note that the series is actually absolutely convergent, so we do know that it converges. Is it equal to x^2 for $x \in [-\pi, \pi]$? At this point, we do not know. Obviously there's still some theory that we're missing. Drawing the graphs for the partial sums, we find that the Fourier series seems to converge to x^2 (periodically extended). Note that there seems to be nothing of that squiggly behavior we saw when drawing the partial sums for $\operatorname{sgn}(x)$. Why not?



8.1 Even/Odd Extensions

Suppose that we have a function $u\colon [0, \pi] \to \mathbf{C}$. We define the even extension u_{e} of u by

$$u_{\mathrm{e}}(x) = \begin{cases} u(x), & 0 \le x \le \pi, \\ u(-x), & -\pi \le x < 0, \end{cases}$$

and the odd extension u_{o} of u by

$$u_{\mathrm{o}}(x) = \begin{cases} u(x), & 0 < x \le \pi, \\ 0, & x = 0, \\ -u(-x), & -\pi \le x < 0, \end{cases}$$

So note that we only have a function defined on half the interval $[-\pi, \pi]$ and that we extend this to the other half. Since we now obtain an odd or even function (depending on choice), we find that the Fourier series will contain only sine or cosine terms. We call this the **sine series** or **cosine series** for a function $u \in L^2(0, \pi)$.

9 What if $T \neq 2\pi$?

As stated earlier, it's not a problem to use functions with a different period than 2π. For this purpose, if u is a T-periodic function, we define

$$\Omega = \frac{2\pi}{T}.$$

The real Fourier series of u is then given by

$$u(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\Omega x + b_k \sin k\Omega x,$$

where

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} u(x)\cos k\Omega x\,dx \quad \text{and} \quad b_k = \frac{2}{T}\int_{-T/2}^{T/2} u(x)\sin k\Omega x\,dx.$$

The complex series is given by

$$u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ik\Omega x}, \quad \text{where } c_k = \frac{1}{T}\int_{-T/2}^{T/2} u(x)e^{-ik\Omega x}\,dx.$$

Example

Find the Fourier series of $u(x) = |x|$, $-1 \le x \le 1$.

Solution. We consider the periodic extension of the function u with the period $T = 2$. Then $\Omega = 2\pi/2 = \pi$ and for $k \neq 0$,

$$\begin{aligned}
c_k &= \frac{1}{2}\int_{-1}^{1} |x|e^{-ik\pi x}\,dx = \frac{1}{2}\int_{-1}^{0} -xe^{-ik\pi x}\,dx + \frac{1}{2}\int_{0}^{1} xe^{-ik\pi x}\,dx \\
&= \frac{1}{2}\left(\left[\frac{-xe^{-ik\pi x}}{-ik\pi}\right]_{-1}^{0} + \int_{-1}^{0}\frac{e^{-ik\pi x}}{-ik\pi}\,dx\right) + \frac{1}{2}\left(\left[\frac{xe^{-ik\pi x}}{-ik\pi}\right]_{0}^{1} - \int_{0}^{1}\frac{e^{-ik\pi x}}{-ik\pi}\,dx\right) \\
&= \frac{1}{2}\left(\frac{e^{ik\pi}}{ik\pi} + \left[\frac{e^{-ik\pi x}}{-k^2\pi^2}\right]_{-1}^{0}\right) + \frac{1}{2}\left(-\frac{e^{-ik\pi}}{ik\pi} - \left[\frac{e^{-ik\pi x}}{-k^2\pi^2}\right]_{0}^{1}\right) \\
&= \frac{1}{2}\left(-\frac{1}{k^2\pi^2} + \frac{e^{ik\pi}}{k^2\pi^2} + \frac{e^{-ik\pi}}{k^2\pi^2} - \frac{1}{k^2\pi^2}\right) = \frac{(-1)^k - 1}{k^2\pi^2}.
\end{aligned}$$

For $k = 0$,

$$c_0 = \frac{1}{2}\int_{-1}^{1} |x|\,dx = \int_{0}^{1} x\,dx = \frac{1}{2}.$$

Hence

$$\begin{aligned}
u(x) &\sim \frac{1}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^k - 1}{k^2\pi^2} e^{ik\pi x} \\
&= \frac{1}{2} + \sum_{k=-\infty}^{-1} \frac{(-1)^k - 1}{k^2\pi^2} e^{ik\pi x} + \sum_{k=1}^{\infty} \frac{(-1)^k - 1}{k^2\pi^2} e^{ik\pi x} \\
&= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{-k} - 1}{(-k)^2\pi^2} e^{-ik\pi x} + \sum_{k=1}^{\infty} \frac{(-1)^k - 1}{k^2\pi^2} e^{ik\pi x} \\
&= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k - 1}{k^2\pi^2}\left(e^{ik\pi x} + e^{-ik\pi x}\right) = \frac{1}{2} + 2\sum_{k=1}^{\infty} \frac{(-1)^k - 1}{k^2\pi^2}\cos(k\pi x) \\
&= \frac{1}{2} - 4\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2\pi^2}\cos((2k+1)\pi x),
\end{aligned}$$

where we used Euler's formulas and the fact that $c_{-k} = c_k$ and finally that $c_{2k} = 0$ for $k \in \mathbf{Z}$ with $k \neq 0$.