

Lecture 2: Linear Algebra, Infinite Dimensional Spaces and Functional Analysis

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“You have no respect for logic. I have no respect for those who have no respect for logic.”
—Julius Benedict

1 Remember Linear algebra? Finite Dimensional Spaces

Let V be a **linear space** (sometimes we say vector space) over the complex (sometimes real) numbers. We recall some definitions from linear algebra. Elements of a linear space can be added and multiplied by constants and still belong to the linear space (this is the linearity):

$$u, v \in V \quad \Rightarrow \quad \alpha u + \beta v \in V, \quad \alpha, \beta \in \mathbf{C} \text{ (or } \mathbf{R}\text{)}.$$

The operations addition and multiplication by constant behaves like we expect (associative, distributive and commutative). Multiplication of vectors is not defined in general, but as we shall see we can define different useful products in many cases.



Linear combination

Definition. Let $u_1, u_2, \dots, u_n \in V$. We call

$$u = \sum_{k=1}^n \alpha_k u_k = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$$

a **linear combination**. If

$$\sum_{k=1}^n \alpha_k u_k = 0 \quad \Leftrightarrow \quad \alpha_1 = \alpha_2 = \dots = \alpha_n = 0,$$

we say that u_1, u_2, \dots, u_n are **linearly independent**. The **linear span** $\text{span}\{u_1, u_2, \dots, u_n\}$ of the vectors u_1, u_2, \dots, u_n is defined as the set of all linear combinations of these vectors (which is a linear space).

You’ve seen plenty of linear spaces before. One such example is the euclidian space \mathbf{R}^n consisting of elements (x_1, x_2, \dots, x_n) , where $x_i \in \mathbf{R}$. Recall also that you’ve seen linear spaces that consisted of polynomials. The fact that our definition is general enough to cover many cases will prove to be very fruitful.



Basis

Definition. A subset $\{v_1, v_2, \dots, v_n\} \subset V$ of linearly independent vectors is called a **base** for V if $V = \text{span}\{v_1, v_2, \dots, v_n\}$ (meaning that every vector $v \in V$ can be expressed uniquely as a linear combination of the elements v_1, v_2, \dots, v_n). The non-negative integer n is called the **dimension** of V : $\dim(V) = n$.

In general, however, we do not wish to restrict ourselves to finite dimensions or vectors of complex numbers.

2 Sequences

We denote a sequence u_1, u_2, u_3, \dots (or u_1, u_2, \dots, u_n if it is a finite sequence) of elements of a linear space V by $(u_k)_{k=1}^{\infty}$ ($(u_k)_{k=1}^n$). If there's no risk of misunderstanding, we might just say "the sequence u_n ."

2.1 Infinite Sequences

As an example, consider the sequence $u_n = x + \frac{1}{n}$ in \mathbf{R} . That means that

$$u_1 = x + 1, \quad u_2 = x + \frac{1}{2}, \quad u_3 = x + \frac{1}{3}, \quad \dots$$

We see that as $n \rightarrow \infty$, clearly $u_n \rightarrow x$. In other words, the sequence u_n *converges* to x . Reformulating a bit, what we have is that

$$\lim_{n \rightarrow \infty} |u_n - x| = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Why write in this particular manner? We'll see... All of this probably feels natural in this setting, but we will generalize this to have meaning for other linear spaces than \mathbf{R} .

Another example could be the decimal expansion of a real number, say

$$x_1 = 3, \quad x_2 = 3.1, \quad x_3 = 3.14, \quad x_4 = 3.141, \quad x_5 = 3.1415, \quad \dots,$$

where we might suspect that $x_n \rightarrow \pi$ as $n \rightarrow \infty$.

We've seen other examples of sequences in the form of partial sums like, for example,

$$S_n = \sum_{k=0}^n \left(\frac{1}{2}\right)^k, \quad n = 0, 1, 2, \dots,$$

where

$$S_n = \frac{1 - (1/2)^{n+1}}{1 - 1/2} = 2(1 - 2^{-n-1}) \rightarrow 2 \text{ as } n \rightarrow \infty.$$

This was the reason for writing that $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2$.

Sequences do not need to be convergent. If $x_n = n$, $n = 0, 1, 2, \dots$, for example, we obtain that

$$x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad \dots,$$

so clearly $x_n \rightarrow \infty$. We might also consider something like $x_n = (-1)^n$ so that

$$x_0 = 1, \quad x_1 = -1, \quad x_2 = 1, \quad x_3 = -1, \quad \dots,$$

where it is clear that x_n has no limit as $n \rightarrow \infty$ (but the sequence is bounded).

2.2 Sequences of Functions

We've already seen the example $u_n = x + 1/n$, which is a sequence of functions (with the variable x). Another example of a sequence of functions is

$$u_n(x) = n \left(\sin \left(x + \frac{1}{n} \right) - \sin(x) \right), \quad n = 1, 2, 3, \dots$$

What happens as $n \rightarrow \infty$?

Another example would be $u_n(x) = \exp(nx)$, $n = 0, 1, 2, \dots$. As $n \rightarrow \infty$, we find that

$$u_n(x) \rightarrow \begin{cases} \infty, & x > 0 \\ 1, & x = 0, \\ 0, & x < 0. \end{cases}$$

We've also worked with function series previously (and obviously last lecture). The power series from TATA42 are an example:

$$S_n(x) = c_0 + c_1x + c_2x^2 + \dots + c_nx^n, \quad n = 0, 1, 2, \dots,$$

where we wrote

$$S(x) = \sum_{k=0}^{\infty} c_k x^k = \lim_{n \rightarrow \infty} S_n(x)$$

for those x where the series converges. That is, we *define* $S(x)$ as the limit of the sequence $S_n(x)$ as $n \rightarrow \infty$ whenever this limit exists.

Yet another example (last one, I promise..) is the partial Fourier sums we saw last lecture:

$$S_n(x) = \sum_{k=-n}^n c_k e^{ikx},$$

where c_k are the Fourier coefficients of some function $u: [-\pi, \pi] \rightarrow \mathbf{C}$. The main problem in the first half of this course is dealing with the sequence S_n of partial Fourier sums. In particular, we're interested in whether $S_n(x)$ converges (in some sense) and if the limit is actually $u(x)$ or not.

2.3 Continuity

Sequences are often useful when speaking of continuity of a function u from subsets of \mathbf{R} or \mathbf{C} into \mathbf{R} or \mathbf{C} . I'm thinking of Heine's definition of continuity:

$$u \text{ is continuous at } x \iff \lim_{n \rightarrow \infty} u(x_n) = u(x) \text{ for any sequence } x_n \text{ such that } \lim_{n \rightarrow \infty} x_n = x,$$

where we require that all x_n belongs to the domain of u .

3 Normed Linear Spaces

To measure distances between elements in a linear space (or "lengths" of elements), we define the abstract notion of a **norm** on a linear space (in the cases where this is allowed).



Norm

Definition. A normed linear space is a linear space V endowed with a norm $\| \cdot \|$ that assigns a non-negative number to each element in V in a way such that

- (i) $\|u\| \geq 0$ for every $u \in V$,
- (ii) $\|\alpha u\| = |\alpha| \|u\|$ for $u \in V$ and every constant α ,
- (iii) $\|u + v\| \leq \|u\| + \|v\|$ for every $u, v \in V$.

We note that in linear algebra, we typically used the norm $|\cdot|$ on the euclidean space \mathbf{R}^n (or \mathbf{C}^n). We will use different types of norms in this course since we will be dealing with more complex spaces.

An element e in V with length 1, that is, $\|e\| = 1$, is called a **unit vector**.



Some examples of normed spaces

- (i) The space \mathbf{R}^n with the norm $\|(x_1, x_2, \dots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$.
- (ii) The space \mathbf{R}^n with the norm $\|x\| = \max\{|x_1|, |x_2|, \dots, |x_n|\}$.

The first example is obviously already something you're familiar with. It is also an example of something we will call an inner product space below. The second example is a bit different. In some sense equivalent, but the norms yield different values for the same vector. Try to prove that the second one satisfies all the requirements for a norm.



The space of continuous functions with sup-norm

The space $C[a, b]$ consisting of continuous functions on the closed interval $[a, b]$ endowed with the norm

$$\|f\|_{C[a,b]} = \max_{a \leq t \leq b} |f(t)|, \quad f \in C[a, b].$$



Sequence spaces

The space l^1 consisting of all sequences (x_1, x_2, x_3, \dots) such that the norm

$$\|x\|_{l^1} = \sum_{k=1}^{\infty} |x_k| < \infty.$$

We might also consider the space l^p for $1 \leq p < \infty$ with the norm

$$\|x\|_{l^p} = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} < \infty.$$



The space of absolutely integrable functions

The space $L^1(\mathbf{R})$ of all integrable (on \mathbf{R}) functions with the norm

$$\|f\|_{L^1(\mathbf{R})} = \int_{-\infty}^{\infty} |f(x)| dx.$$

In other words, all functions that are absolutely integrable on \mathbf{R} . Note here that there's an army of dogs buried here. Indeed, the integral is not in the sense we're used to but rather in the form of the Lebesgue integral. We will not get stuck at this point, but it might be good to know.

Exercise: Prove that the spaces above are normed linear spaces. Do you see any useful ways to consider some "multiplication" of vectors?

We see that an underlying linear space (like \mathbf{R}^n) might be endowed with different norms. This is true in general, and changing norms usually changes the results (at least for infinite dimensional spaces).

4 Convergence in Normed Spaces

Let u_1, u_2, \dots be a sequence in a normed space V . We say that $u_n \rightarrow u$ for some $u \in V$ if $\|u_n - u\| \rightarrow 0$ as $n \rightarrow \infty$. This is called **strong convergence** or **convergence in norm**. Note that we assumed above that the element u belonged to V . This may not be the case for every convergent sequence. How? Typically, we consider a linear subspace W of V . A sequence in W might be convergent when viewed as a sequence in V , but the limiting element might not belong to W .

For this course, we will mainly study the space E which consists of piecewise continuous functions. This will ensure that some things are easy, but unfortunately the space E with the norms and inner products we are interested in will have some issues (E is neither complete nor closed, whatever that means.). This will not be a big problem for us, but it's worth mentioning if we wish to do Fourier analysis in a more general setting.

Analogously with real analysis, we can define continuous mappings on normed spaces.



Continuity in normed spaces

Definition. Let V and W be normed spaces. A function $u: V \rightarrow W$ is said to be continuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$x, y \in V, \|x - y\|_V < \delta \quad \Rightarrow \quad \|u(x) - u(y)\|_W < \epsilon.$$

Basically this states that we call the function $u: V \rightarrow W$ continuous if x and y are close in V implies that $u(x)$ and $u(y)$ are close in W .

4.1 Series in Normed Spaces

Let u_1, u_2, u_3, \dots be a sequence in V . How do we interpret an expression of the form

$$S = \sum_{k=1}^{\infty} u_k, \tag{1}$$

that is, what does an infinite sum of elements in V mean? We define the partial sums by

$$S_n = \sum_{k=1}^n u_k, \quad n = 1, 2, 3, \dots$$

If S_n converges to some $S \in V$ in norm, that is,

$$\lim_{n \rightarrow \infty} \left\| S - \sum_{k=1}^n u_k \right\| = 0,$$

then we write that (1) is convergent. Notice that this does *not* mean that

$$\sum_{k=1}^{\infty} \|u_k\| < \infty.$$

If this second series of real numbers is convergent, we call (1) **absolutely convergent** (compare with what we did in TATA42). Note also that an absolutely convergent series is convergent in the sense above (why?).

5 Inner Product Spaces

A norm is not enough to define a suitable geometry for our purposes, so we will usually work with inner product spaces instead.



Inner product

Definition. An inner product $\langle \cdot, \cdot \rangle$ on a vector space V is a complex valued (sometimes real) function on $V \times V$ such that

(i) $\langle u, v \rangle = \overline{\langle v, u \rangle}$

(ii) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

(iii) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$

(iv) $\langle u, u \rangle \geq 0$

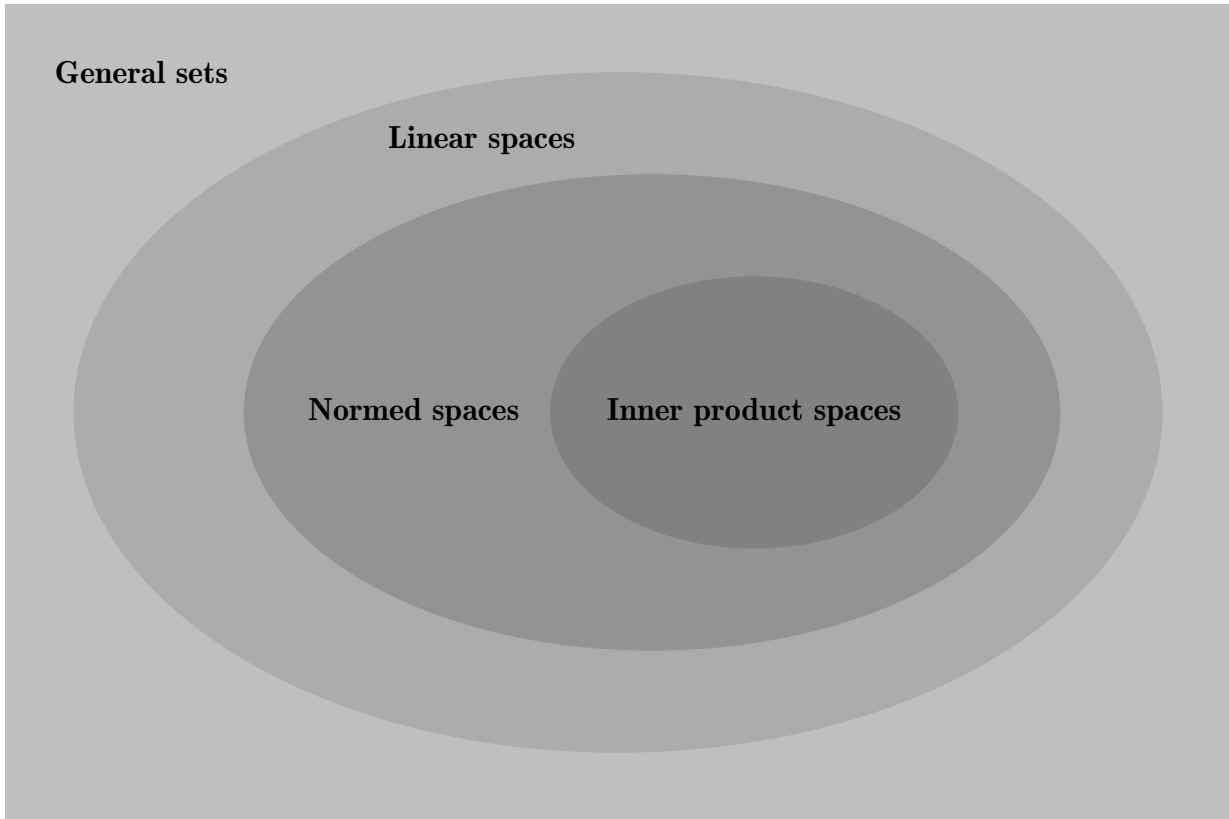
(v) $\langle u, u \rangle = 0$ if and only if $u = 0$.

Note that (i) and (ii) implies that $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ and that (i) and (iii) implies that $\langle u, \alpha v \rangle = \overline{\alpha} \langle u, v \rangle$.

In an inner product space, we use $\|u\| = \sqrt{\langle u, u \rangle}$ as the norm. Why is this a norm? We'll get to that.



Notice that if we're given a linear space of functions, there's an infinite number of different inner products on this space that provides the same *geometry*. Suppose that $\langle u, v \rangle$ is an inner product. Then $\alpha \langle u, v \rangle$ is also an inner product for any $\alpha > 0$.



The inner product space \mathbf{C}^n

Definition. The space \mathbf{C}^n consisting of n -tuples (z_1, z_2, \dots, z_n) with

$$\langle z, w \rangle = \sum_{k=1}^n z_k \overline{w_k}, \quad z, w \in \mathbf{C}^n,$$

is an inner product space.



The inner product space l^2

Definition. The space l^2 consisting of all sequences (x_1, x_2, x_3, \dots) of complex numbers such that the norm

$$\|x\|_{l^2} = \left(\sum_{k=1}^{\infty} |x_k|^2 \right)^{1/2} < \infty.$$

This is an inner product space if

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}, \quad x, y \in l^2.$$



The inner product space $L^2(a, b)$

Definition. The space $L^2(a, b)$ consists of all “square integrable” functions with the inner product

$$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)} dt.$$

Note that $a = -\infty$ and/or $b = \infty$ is allowed.

Why not the same examples as for the normed spaces? The simple answer is that most of those examples are *not* inner product spaces. The last two examples above are very important and the fact that it’s the number 2 is not random and this is actually the only choice for when $L^p(a, b)$, which consists of functions for which

$$\|f\|_{L^p(a,b)} = \left(\int_a^b |f(t)|^p dt \right)^{1/p} < \infty$$

are inner product spaces. Again, we also note that the integrals above are more general than what we’ve seen earlier but if the function f is nice enough the value will coincide with the (generalized) Riemann integral.



Orthogonality

Definition. If $u, v \in V$ and V is an inner product space, we say that u and v are orthogonal if $\langle u, v \rangle = 0$. We denote this by $u \perp v$.

A sequence u_n is called **pairwise orthogonal** if $\langle u_i, u_j \rangle = 0$ for every $i \neq j$. For sequences of this type, we have the generalized Pythagorean theorem.



Theorem. If u_1, u_2, \dots, u_n are pairwise orthogonal, then

$$\|u_1 + u_2 + \dots + u_n\|^2 = \|u_1\|^2 + \|u_2\|^2 + \dots + \|u_n\|^2.$$



The Cauchy-Schwarz inequality

Theorem. If $u, v \in V$ and V is an inner product space, then $|\langle u, v \rangle| \leq \|u\|\|v\|$.

Proof. Assume that $v \neq 0$ (the inequality is trivial if $v = 0$) and define $\lambda = \langle u, v \rangle / \|v\|^2$. Then

$$\begin{aligned} \|u - \lambda v\|^2 &= \langle u - \lambda v, u - \lambda v \rangle = \|u\|^2 - \bar{\lambda} \langle u, v \rangle - \lambda \langle v, u \rangle + |\lambda|^2 \|v\|^2 \\ &= \|u\|^2 - \bar{\lambda} \langle u, v \rangle - \lambda \overline{\langle u, v \rangle} + |\lambda|^2 \|v\|^2 \\ &= \|u\|^2 - \frac{|\langle u, v \rangle|^2}{\|v\|^2} - \frac{|\langle u, v \rangle|^2}{\|v\|^2} + \frac{|\langle u, v \rangle|^2}{\|v\|^2} = \|u\|^2 - \frac{|\langle u, v \rangle|^2}{\|v\|^2} \end{aligned}$$

so

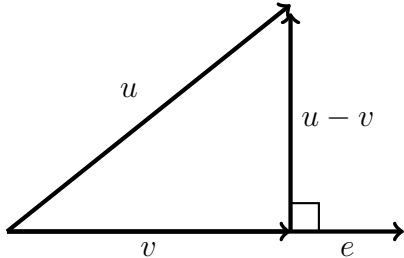
$$0 \leq \|u - \lambda v\|^2 = \|u\|^2 - \frac{|\langle u, v \rangle|^2}{\|v\|^2} \Leftrightarrow |\langle u, v \rangle|^2 \leq \|u\|^2 \|v\|^2,$$

which implies the inequality. □

6 Orthogonal Projection

Let $e \in V$ with $\|e\| = 1$. For $u \in V$, we define the **orthogonal projection** v of u on e by $v = \langle u, e \rangle e$. This is reasonable since $u - v \perp e$:

$$\langle u - v, e \rangle = \langle u, e \rangle - \langle v, e \rangle = \langle u, e \rangle - \langle u, e \rangle \langle e, e \rangle = 0.$$



Note that

$$\begin{aligned} \|u\|^2 &= \|u - v + v\|^2 \\ &= \|u - v\|^2 + \|v\|^2 \\ &= \|u - v\|^2 + |\langle u, e \rangle|^2. \end{aligned}$$



ON system

Definition. Let V be an inner product space. We call

- (i) $\{e_1, e_2, \dots, e_n\} \subset V$,
- (ii) or $\{e_1, e_2, \dots\} \subset V$,

an ON system in V if $e_i \perp e_j$ for $i \neq j$ and $\|e_i\| = 1$ for all i .

We do *not* assume that V is finite dimensional and that n is the dimension, and we do *not* assume that the ON system consists of finitely many elements.

If the ON system is finite, consider $W = \text{span}\{e_1, e_2, \dots, e_n\} \subset V$. We define the orthogonal projection Pv of a vector $v \in V$ onto the linear space W by

$$Pv = \sum_{k=1}^n \langle v, e_k \rangle e_k.$$

If $v \in W$, then clearly $Pv = v$. If $v \notin W$, then Pv is the vector that minimizes $\|v - Pv\|$. Note that this happens if $v - Pv \perp W$ (meaning perpendicular to every vector in W). We also note that

$$\|v\|^2 = \|v - Pv\|^2 + \sum_{k=1}^n |\langle v, e_k \rangle|^2$$

These facts are well-known from linear algebra.

If the ON system is infinite, let

$$P_n v = \sum_{k=1}^n \langle v, e_k \rangle e_k, \quad v \in V, \quad n = 1, 2, 3, \dots$$

Each $P_n v$ is the projection on a specific n -dimensional subspace of V (the order of the elements in the ON system is fixed). Let us state the famous Bessel's inequality.



Bessel's inequality

Theorem. Let V be an inner product space, let $v \in V$ and let $\{e_1, e_2, \dots\}$ be an ON system in V . Then

$$\sum_{k=1}^{\infty} |\langle v, e_k \rangle|^2 \leq \|v\|^2.$$

Since $\|v\| < \infty$ for every $v \in V$, this inequality proves that the series in the left-hand side converges. A direct consequence of this is the Riemann-Lebesgue lemma.



The Riemann-Lebesgue Lemma

Theorem. Let V be an inner product space, let $v \in V$ and let $\{e_1, e_2, \dots\}$ be an ON system in V . Then

$$\lim_{n \rightarrow \infty} \langle v, e_n \rangle = 0.$$

6.1 The Infinite Dimensional Case

If $\dim(V) = n$ and our ON system has n elements, then we know that we can always represent $v \in V$ as $v = \sum_{k=1}^n \langle v, e_k \rangle e_k$ (standard linear algebra). What happens if $\dim(V) = \infty$? When can we expect that an infinite ON system allows for something similar?



Closed ON systems

Definition. Let V be an inner product space with $\dim(V) = \infty$. We call an orthonormal system $\{e_1, e_2, \dots\} \subset V$ **closed** if for every $v \in V$ and every $\epsilon > 0$, there exists a sequence c_1, c_2, \dots, c_n of constants such that

$$\left\| v - \sum_{k=1}^n c_k e_k \right\| < \epsilon. \tag{2}$$

How do we typically find numbers c_k that work (they're not unique)? One answer comes in the form of orthogonal projections.



Fourier coefficients

Definition. For a given ON system, the complex numbers $\langle v, e_k \rangle$, $k = 1, 2, \dots$, are called the **generalized Fourier coefficients** of v .

We define the operator P_n that projects a vector onto the linear space spanned by $\{e_1, e_2, \dots, e_n\}$ by

$$P_n v = \sum_{k=1}^n \langle v, e_k \rangle e_k, \quad v \in V.$$

We now note that the choice $c_k = \langle v, e_k \rangle$ is the choice that minimizes the left-hand side in (2). Indeed, suppose that $u = \sum_{k=1}^n c_k e_k$ for some constants c_k . Then

$$\begin{aligned} \|v - u\|^2 &= \|v - P_n v + P_n v - u\|^2 = \|(v - P_n v) \perp (P_n v - u)\|^2 = \|v - P_n v\|^2 + \|P_n v - u\|^2 \\ &= \|v - P_n v\|^2 + \left\| \sum_{k=1}^n (\langle v, e_k \rangle - c_k) e_k \right\|^2 \\ &= \|v - P_n v\|^2 + \sum_{k=1}^n |\langle v, e_k \rangle - c_k|^2, \end{aligned}$$

so obviously $c_k = \langle v, e_k \rangle$ is the unique choice that minimizes $\|v - u\|$. In other words, $u = P_n v$ is the only element that minimizes $\|v - u\|$.

Because of this, one can reformulate (equivalently) the definition of a closed ON system as follows.



Definition. Let V be an inner product space with $\dim(V) = \infty$. We call an orthonormal system $\{e_1, e_2, \dots\} \subset V$ **closed** if for every $v \in V$

$$\lim_{n \rightarrow \infty} \left\| v - \sum_{k=1}^n \langle v, e_k \rangle e_k \right\| = 0.$$

We note that in the case where the ON system is closed, we can strengthen Bessel's inequality (by replacing the inequality with equality) obtaining what is known as Parseval's identity (or Parseval's formula). As it turns out, the fact that Parseval's identity holds for an ON-system is equivalent to the fact that the ON-system is closed.



Theorem. Suppose that $W = \{e_1, e_2, \dots\}$ is an ON system for the inner product space V . Then W is closed if and only if **Parseval's identity** holds:

$$\sum_{k=1}^{\infty} |\langle v, e_k \rangle|^2 = \|v\|^2$$

for every $v \in V$.

Proof. Let $v \in V$. Then

$$\|v\|^2 = \|v - P_n v\|^2 + \|P_n v\|^2$$

since $v - P_n v \perp P_n v$. Hence

$$\left\| v - \sum_{k=1}^n \langle v, e_k \rangle e_k \right\|^2 = \|v\|^2 - \sum_{k=1}^n |\langle v, e_k \rangle|^2$$

and letting $n \rightarrow \infty$ in this equality, we see that closedness is equivalent with Parseval's identity holding. \square



Definition. An ON-system $\{e_1, e_2, \dots\}$ in V is called complete if, for every $v \in V$,

$$\langle v, e_k \rangle = 0 \text{ for all } k = 1, 2, 3, \dots \Leftrightarrow v = 0.$$

We realize that completeness is something we want if we wish to use an ON-system as a *basis* for V since we would prefer that representations in terms of linear combinations of basis vectors to be unique.



Generalized Parseval's identity

Theorem. Suppose that $\{e_1, e_2, e_3, \dots\}$ is a closed infinite ON-system in V and let $u, v \in V$. If $a_k = \langle u, e_k \rangle$ and $b_k = \langle v, e_k \rangle$, then

$$\langle u, v \rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}.$$

Proof. Since V is a complex inner product space, the following equality (usually known as the polarization identity) holds:

$$\langle u, v \rangle = \frac{1}{4} (\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2).$$

Since we have a closed ON-system, Parseval's formula holds, so it is clear that

$$\|u + v\|^2 = \sum_{k=1}^{\infty} |a_k + b_k|^2$$

since $\langle u + v, e_k \rangle = \langle u, e_k \rangle + \langle v, e_k \rangle = a_k + b_k$. Similarly, we obtain that

$$\|u - v\|^2 = \sum_{k=1}^{\infty} |a_k - b_k|^2, \quad \|u + iv\|^2 = \sum_{k=1}^{\infty} |a_k + ib_k|^2, \quad \|u - iv\|^2 = \sum_{k=1}^{\infty} |a_k - ib_k|^2.$$

Note also that (verify this directly)

$$a_k \overline{b_k} = \frac{1}{4} (|a_k + b_k|^2 - |a_k - b_k|^2 + i|a_k + ib_k|^2 - i|a_k - ib_k|^2),$$

so the identity in the theorem must hold. □

7 Fourier Series?

So that brings us back to one of the main subjects of this course: Fourier series. Let's look at a particular inner product space.

7.1 The ON Systems

We consider the space $L^2(-\pi, \pi)$ consisting of square integrable functions $u: [-\pi, \pi] \rightarrow \mathbf{C}$:

$$\int_{-\pi}^{\pi} |u(x)|^2 dx < \infty.$$

We define the inner product on this space by

$$\langle u, v \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) \overline{v(x)} dx.$$

Note that this infers that we have the norm

$$\|u\| = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(x)|^2 dx \right)^{1/2},$$

which by definition is finite for $u \in L^2(-\pi, \pi)$. Let's consider two special orthonormal systems in this space.

The set of functions e^{ikx} , $k \in \mathbf{Z}$, is a closed orthonormal system in E with the inner product defined above. We consider E as a subspace of $L^2(-\pi, \pi)$. Clearly we have

$$\|e^{ikx}\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-ikx} dx = \frac{2\pi}{2\pi} = 1.$$

Similarly, if $k, l \in \mathbf{Z}$ and $k \neq l$, we have

$$\langle e^{ikx}, e^{ilx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-ilx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-l)x} dx = 0$$

since $e^{i(k-l)x}$ is 2π -periodic. So this is an ON-system in E . The fact that it is closed is a more difficult argument so we'll get back to this on lecture 5. Note though, that E is a subspace of $L^2(-\pi, \pi)$ that has some issues. For example there are sequences in E that converge (in the L^2 -norm) to elements outside of E . This is a disadvantage, but nothing that will cause too many problems for us.

7.1.1 The Real System

The set of functions $\frac{1}{\sqrt{2}}, \cos kx, k = 1, 2, 3, \dots, \sin kx, k = 1, 2, 3, \dots$, is a closed orthonormal system in E with the inner product

$$\langle u, v \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) \overline{v(x)} dx.$$

Note that the *normalization constant* is different compared to the complex case (why do you think that is?). We should observe that these two systems are equivalent due to Euler's formulas.

8 The Space E as an Inner Product Space

So the question right now is *what do we really need?*

Most of the results we're going to see have a more general and complete version, but we would need considerably more time to develop the necessary tools to attack these problems. So what we're going to do instead is to consider the space E with the inner product

$$\langle u, v \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) \overline{v(x)} dx, \quad u, v \in E. \quad (3)$$

As stated above, this space has some serious drawbacks, but these problems are not crucial to what we're going to do.

First, let's verify that things work as expected. When we write E , we now mean the combination of the set E of piecewise continuous functions combined with the inner product defined by (3).

(i) *E is a linear space.* Obviously, if $u \in E$ and α is a constant, then αu has the same exception points as u (unless $\alpha = 0$) and the right- and lefthand limits will exist for $\alpha u(x)$. Let $u, v \in E$ and let a_1, a_2, \dots, a_n be the exceptions points of u and b_1, b_2, \dots, b_m be the exception points of v . Then $u + v$ has (at most) $m + n$ exception points. Indeed, if we sort the exception points as $c_1 < c_2 < \dots < c_{n+m}$, then $u + v$ will be continuous on each $]c_i, c_{i+1}[$ and the right- and lefthand limits at the exception points will exist since either it is an exception point for u or v (potentially both), or it is a point of continuity for u or v . Therefore the limit of the sum exist.

(ii) *Equation (3) defines an inner product on E .* Most of the properties follow from the linearity of the integral. The fact that $\langle u, u \rangle = 0$ implies that $u = 0$ is clear since

$$\langle u, u \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(x)|^2 dx = 0$$

so $u = 0$ is the only possible piecewise continuous function (if $u(x_0) \neq 0$ at some point then there is an interval $]x_0 - \delta, x_0 + \delta[$ where $|u(x)| > 0$ and so the Riemann integral will be strictly greater than zero).

8.1 Fourier Coefficients and the Riemann Lebesgue Lemma

So in general, we know that $\langle u, e_k \rangle \rightarrow 0$ as $k \rightarrow \infty$ if $\{e_1, e_2, \dots\}$ is an ON system with respect to the inner product at hand (in our case (3)). This was a consequence of Bessel's inequality. In particular, this means that for $u \in E$, we have

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} u(x) e^{-inx} dx = 0.$$

Note that this implies that

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} u(x) \sin(nx) dx = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} u(x) \cos(nx) dx = 0.$$

So apparently these limits hold for all piecewise continuous functions. However, these identities are also true for $u \in L^1(-\pi, \pi)$ (this needs a proof however).

8.2 Bessel's Inequality Turned Parseval's Identity

Taking for granted that this ON system is closed (which is not clear at all at this point but we'll get back to that), we conclude by noting that Parseval's identity looks like this:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(x)|^2 dx = \sum_{k=-\infty}^{\infty} |c_k|^2,$$

where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) e^{-ikx} dx, \quad k \in \mathbf{Z}.$$

The general form is given by

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) \overline{v(x)} dx = \sum_{k=-\infty}^{\infty} c_k \overline{d_k},$$

where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) e^{-ikx} dx \quad \text{and} \quad d_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(x) e^{-ikx} dx, \quad k \in \mathbf{Z}.$$

9 Why is $\sqrt{\langle u, u \rangle}$ a Norm?

Let's define $\|u\| = \sqrt{\langle u, u \rangle}$ for $u \in V$. Then clearly $\|u\| \geq 0$ and $\|u\| = 0$ if and only if $u = 0$ (since this holds for the inner product). Furthermore, if $\alpha \in \mathbf{C}$ we have

$$\|\alpha u\| = \sqrt{\langle \alpha u, \alpha u \rangle} = \sqrt{\alpha \overline{\alpha} \langle u, u \rangle} = |\alpha| \sqrt{\langle u, u \rangle} = |\alpha| \|u\|.$$

To prove that $\|u + v\| \leq \|u\| + \|v\|$, we note that

$$\|u + v\|^2 = \langle u + v, u + v \rangle = \|u\|^2 + \|v\|^2 + \langle u, v \rangle + \langle v, u \rangle = \|u\|^2 + \|v\|^2 + 2 \operatorname{Re} \langle u, v \rangle.$$

Since $\operatorname{Re} z \leq |z|$ for any $z \in \mathbf{C}$ (why?), the Cauchy-Schwarz inequality implies that

$$2 \operatorname{Re} \langle u, v \rangle \leq 2 |\langle u, v \rangle| \leq 2 \|u\| \|v\|.$$

Thus

$$\|u\|^2 + \|v\|^2 + 2 \operatorname{Re} \langle u, v \rangle \leq \|u\|^2 + \|v\|^2 + 2 \|u\| \|v\| = (\|u\| + \|v\|)^2,$$

so

$$\|u + v\|^2 \leq (\|u\| + \|v\|)^2,$$

which proves that the triangle inequality holds.