Lecture 3: Function Series and Convergence

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"Here, stick around!"
—John Matrix

1 Pointwise Convergence

Let $u_1, u_2, u_3, \ldots$ be a sequence of functions $u_k : I \to \mathbf{C}$, where I is some set of real numbers. We've seen pointwise convergence earlier, but let's formulate it more rigorously.

Pointwise convergence

Definition. We say that $u_k \to u$ pointwise on I as $k \to \infty$ if

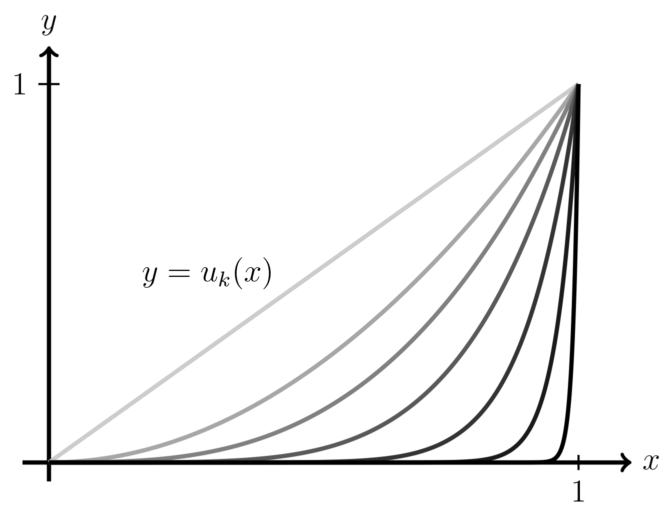
$$\lim_{k\to\infty} u_k(x) = u(x)$$

for every $x \in I$. We often refer to u as the *limiting function*.

Why would this not suffice? Let's consider an example.

Example

Let $u_k(x) = x^k$ if $0 \leq x \leq 1$, $k = 1, 2, 3, \ldots$. Then $u_k(x) \to 0$ for $0 \leq x < 1$ and $u_k(x) \to 1$ when $x = 1$. Clearly u_k is continuous on $[0, 1]$ for every k, but the *limiting function* is discontinuous at $x = 1$.



This is slightly troubling. The fact that certain properties hold for all elements in a sequence but not for the limiting element has caused more than one engineer to assume something dangerous. So can we require something more to ensure that, e.g., continuity is inherited? As we shall see, if the convergence is *uniform* this will be true.

2 Uniform Convergence

Supremum and infimum

Definition. Let $A \subset \mathbf{R}$ be a set of real numbers. Let α be the greatest real number so that $x \geq \alpha$ for every $x \in A$. We call α the **infimum** of A. Let β be the smallest real number so that $x \leq \beta$ for every $x \in A$. We call β the **supremum** of A.

Sometimes the infimum and supremum are called the greatest lower bound and least upper bound instead. Note also that these numbers always exist; see the end of the analysis book (the supremum axiom).

Observe the difference between max/min and sup/inf.

Why is minimum and maximum not enough? Well, consider for example the set $A = [0, 1[$. We see that $\min(A) = 0$ and that this is obviously also the infimum of A. However, there is no maximum element in A. The supremum is equal to the value we would need the maximum to attain, that is $\sup(A) = 1$.

Note though, that if there is a maximum element in A, this will also be the supremum. Similarly, if there is a smallest element in A, this will be the infimum.

So with this in mind, consider the linear space of all functions $f : [a, b] \to \mathbf{C}$. We define a normed space $L^\infty[a, b]$ consisting of those functions which has a finite supremum-norm:

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)| < \infty.$$

This expression always exist. Observe also that $|f(x)| \leq \|f\|_\infty$ for every $x \in [a, b]$. If we were to restrict our attention to continuous functions on $[a, b]$, we could exchange the supremum for maximum since we know that the maximum for a continuous function on a closed interval is attained (see TATA41).

Uniform convergence

Definition. We say that $u_k \to u$ uniformly on $[a, b]$ as $k \to \infty$ if

$$\lim_{k \to \infty} \|u_k - u\|_\infty = 0.$$

Notice that if $u_k \to u$ uniformly on $[a, b]$, then $u_k \to u$ pointwise on $[a, b]$. The converse, however, does not hold. Let's look at the previous example where $u_k = x^k$ for $0 \leq x \leq 1$. Clearly $u_k(x) \to u(x)$ as $k \to \infty$, where $u(x) = 0$ if $0 \leq x < 1$ and $u(1) = 1$. However, the convergence is *not* uniform:

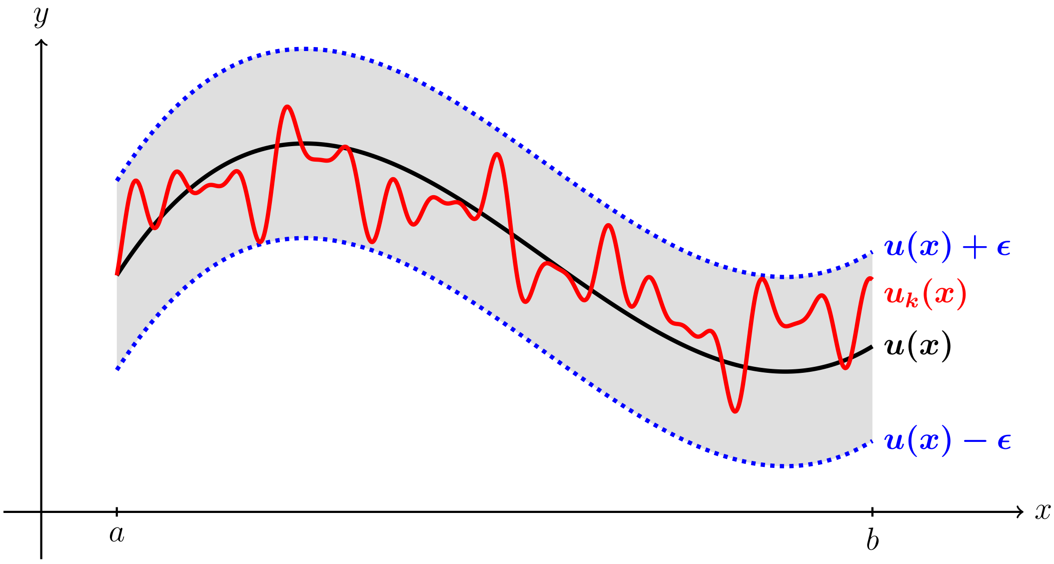
$$|u_k(x) - u(x)| = \begin{cases} x^k, & 0 \leq x < 1, \\ 0, & x = 1, \end{cases} \quad \Rightarrow \quad \|u_k - u\|_\infty = \sup_{0 \leq x < 1} x^k = 1, \quad k = 1, 2, 3, \ldots,$$

so it is not the case that $\|u_k - u\|_\infty$ tends to zero. Therefore the convergence is not uniform. There is another way to see this as well, we'll get to that in the next section when discussing continuity.

By definition, if $u_k \to u$ uniformly on $[a,b]$, this means that for every $\epsilon > 0$, there is some integer N such that

$$k \geq N \quad \Rightarrow \quad \|u_k - u\|_\infty = \sup_{x\in[a,b]} |u_k(x) - u(x)| < \epsilon.$$

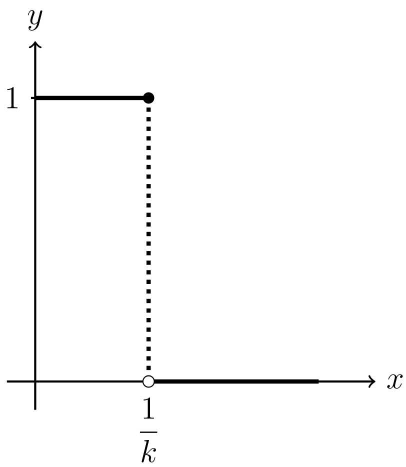
This means that for every $k \geq N$, the difference between $u_k(x)$ and $u(x)$ is less than ϵ *for every* $x \in [a,b]$.



Example

Let $u_k(x) = 0$ if $1/k \leq x \leq 1$ and let $u_k = 1$ if $0 \leq x < 1/k$. Show that $u_k \to u$ pointwise but not uniformly, where $u(x) = 0$ if $x > 0$ and $u(0) = 1$.

Solution. We see that the graph of u_k looks like the figure below.



For any $x \in]0,1]$, it is clear that $u_k(x) = 0$ if $k > 1/x$. So $u_k(x) \to 0$ for any $x \in]0,1]$. For $x = 0$ however, there's no $k > 0$ such that $u_k(0) = 0$. The limiting function is $u(x) = 0$ for $x > 0$ and $u(0) = 1$. Hence the convergence cannot be uniform, similar to the previous example.

Example

Show that $u_k(x) = x + \frac{1}{k}x^2$ converges uniformly on $[0, 2]$.

Solution. Clearly $u_k(x) \to x$ as $k \to \infty$ for $x \in [0, 2]$ (for $x \in \mathbf{R}$ really). Hence the pointwise limit is given by $u(x) = x$. Now, observe that

$$|u_k(x) - u(x)| = \left|\frac{1}{k}x^2\right| \leq \frac{1}{k}x^2,$$

so

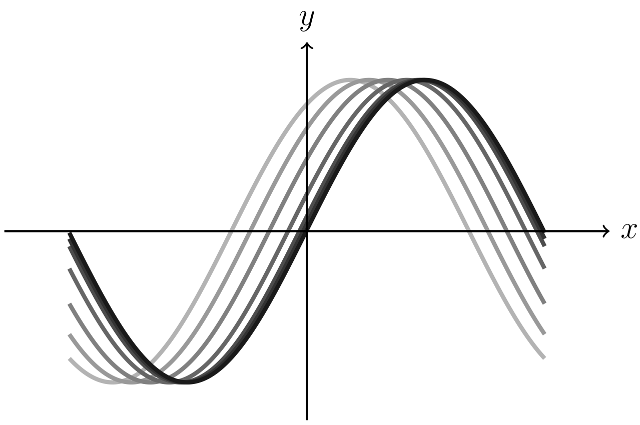
$$\|u_k - u\|_{L^\infty(0,2)} \leq \frac{1}{k}2^2 = \frac{4}{k} \to 0,$$

as $k \to \infty$. Hence the convergence is indeed uniform on $[0, 2]$. Would the convergence be uniform on $\mathbf{R}$?

Example

Let $u_k(x) = \sin(x + 1/k)$ for $-\pi \leq x \leq \pi$ and $k > 0$. Does u_k converge uniformly?

Solution. Since sin is continuous, we have $u_k(x) \to \sin x$ for $x \in \mathbf{R}$.



Since sin is differentiable, the mean value theorem implies that

$$\sin(x + 1/k) - \sin x = (x + 1/k - x)\cos\xi,$$

for some ξ between x and $x + 1/k$. Hence

$$|\sin(x + 1/k) - \sin x| \leq |x + 1/k - x| = \frac{1}{k}$$

since $|\cos\xi| \leq 1$. From this it follows that

$$\sup_x |\sin(x + 1/k) - \sin x| \leq \frac{1}{k} \to 0,$$

so the convergence is uniform.

3 Continuity and Differentiability

Knowing that a sequence u_k converges pointwise to some function u is not enough to infer that properties like continuity and differentiability are inherited. However, uniform convergence implies that certain properties are inherited by the limiting function.

Theorem. If $u_1, u_2, u_3, \ldots$ is a sequence of continuous functions $u_k : [a, b] \to \mathbf{C}$ and $u_k \to u$ uniformly on $[a, b]$, then u is continuous on $[a, b]$.

Proof. To prove that the limiting function u is continuous, we'll need all the δ-ϵ stuff. Let x and x_0 belong to $[a, b]$ and let $\epsilon > 0$ be fixed. We will show that there exists a $\delta > 0$ so that $|u(x) - u(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$, which proves that u is continuous at x_0. Since x_0 is arbitrary, this proves that u is continuous on $[a, b]$.

Now let's do some triangle inequality magic:

$$\begin{aligned}
|u(x) - u(x_0)| &= |u(x) - u_k(x) + u_k(x) - u_k(x_0) + u_k(x_0) - u(x_0)| \\
&\le |u(x) - u_k(x)| + |u_k(x) - u_k(x_0)| + |u_k(x_0) - u(x_0)| \\
&\le \|u - u_k\|_\infty + |u_k(x) - u_k(x_0)| + \|u_k - u\|_\infty = 2\|u - u_k\|_\infty + |u_k(x) - u_k(x_0)|,
\end{aligned}$$

since $|f(x)| \le \|f\|_\infty$ for any $f : [a, b] \to \mathbf{C}$. Since $u_k \to u$ uniformly on $[a, b]$, we know that $\|u_k - u\|_\infty \to 0$, so there exists $N \in \mathbf{N}$ so that $\|u_k - u\|_\infty < \epsilon/3$ for $k \ge N$. Furthermore, since u_k is continuous, there exists a $\delta > 0$ so that $|u_k(x) - u_k(x_0)| < \epsilon/3$ whenever $|x - x_0| < \delta$. Thus we obtain that

$$|u(x) - u(x_0)| < 2 \cdot \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

whenever $|x - x_0| < \delta$. □

Use discontinuity to prove that convergence is *not* uniform

We can exploit the negation of this theorem to prove that a sequence is *not* uniformly convergent. Suppose that

(i) $u_1, u_2, u_3, \ldots$ is a sequence of continuous functions.

(ii) $u_k(x) \to u(x)$ pointwise on $[a, b]$.

(iii) There is some $x_0 \in [a, b]$ where the limiting function u is *not* continuous.

Then the convergence of the sequence can *not* be uniform!

Let's consider the example from the first section again.

Example

Let $u_k(x) = x^k$ if $0 \le x \le 1$, $k = 1, 2, 3, \ldots$. Then $u_k(x) \to 0$ for $0 \le x < 1$ and $u_k(x) \to 1$ when $x = 1$. Clearly u_k is continuous on $[0, 1]$ for every k, but the *limiting* function is discontinuous at $x = 1$. Hence the convergence *cannot* be uniform!

It's not just the continuity that's easier to infer, we can also work with integrals like they were sums and exchange the order of integration and taking limits.

Theorem. Suppose that $u_1, u_2, u_3, \dots$ is a sequence of continuous functions $u_k : [a, b] \to \mathbf{C}$ and that $u_k \to u$ uniformly on $[a, b]$. Then

$$\lim_{k\to\infty} \int_a^b u_k(x)\,dx = \int_a^b \lim_{k\to\infty} u_k(x)\,dx = \int_a^b u(x)\,dx.$$

Proof. Assume that $b > a$. Since the integral is monotonous (we get a bigger value when moving the modulus inside), we see that

$$\begin{aligned}\left|\int_a^b u_k(x)\,dx - \int_a^b u(x)\,dx\right| &= \left|\int_a^b \big(u_k(x) - u(x)\big)\,dx\right| \le \int_a^b \big|u_k(x) - u(x)\big|\,dx \\ &\le \int_a^b \big\|u_k - u\big\|_\infty\,dx = \big\|u_k - u\big\|_\infty \int_a^b dx \\ &= \big\|u_k - u\big\|_\infty (b-a) \to 0, \text{ as } k \to \infty,\end{aligned}$$

since $\|u_k - u\|_\infty$ is independent of x. □

Remark. There are other results of this type with much weaker assumptions. Continuity is not necessary (it is enough it is a sequence of *integrable* functions) and the uniform convergence can be exchanged for weaker types of convergence as well (*dominated convergence*).

Example

Find the value of $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{nx+1}{nx^2+x+n}\,dx$.

Solution. Let $u_n(x) = \dfrac{nx+1}{nx^2+x+n}$, $n = 1, 2, 3, \dots$ and $0 \le x \le 1$. Then

$$\frac{nx+1}{nx^2+x+n} = \frac{x+1/n}{x^2+1+x/n} \to \frac{x}{x^2+1},$$

as $n \to \infty$. Moreover,

$$\begin{aligned}\left|\frac{nx+1}{nx^2+x+n} - \frac{x}{x^2+1}\right| &= \left|\frac{(nx+1)(x^2+1) - x(nx^2+x+n)}{(x^2+1)(nx^2+x+n)}\right| \\ &= \left|\frac{1}{(x^2+1)(nx^2+x+n)}\right| = \frac{1}{n}\left|\frac{1}{(x^2+1)(x^2+x/n+1)}\right| \le \frac{1}{n}\end{aligned}$$

since $1 + x^2 \ge 1$ and $x^2 + x/n + 1 \ge 1$. Clearly this means that

$$\sup_{0\le x\le 1}\left|\frac{nx+1}{nx^2+x+n} - \frac{x}{x^2+1}\right| \le \frac{1}{n} \to 0,$$

as $n \to \infty$. The convergence is therefore uniform and

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 \frac{nx+1}{nx^2+x+n}\,dx &= \int_0^1 \lim_{n\to\infty} \frac{nx+1}{nx^2+x+n}\,dx = \int_0^1 \frac{x}{x^2+1}\,dx \\ &= \left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^1 = \frac{\ln 2}{2}.\end{aligned}$$

Integrals and uniform limits

Notice the steps in the previous example:

(i) Find the pointwise limit $u(x)$ of $u_k(x)$.

(ii) Find a *uniform* bound for $|u_k(x) - u(x)|$ that tends to zero as $k \to \infty$ (independently of x).

(iii) Deduce that $u_k \to u$ uniformly.

(iv) Move the limit inside the integral, effectively replacing $\lim u_k$ by u, and calculate the resulting integral.

There are no short-cuts. Without a clear motivation about the fact that we have uniform convergence and what this means, the result will be zero points (even with the "right answer").

So what about taking derivatives? That's slightly more difficult.

Theorem. Let $u_1, u_2, u_3, \ldots$ be a sequence of differentiable functions $u_k : [a, b] \to \mathbf{C}$. If $u_k \to u$ pointwise on $[a, b]$ and $u_k' \to v$ uniformly on $[a, b]$, where v is continuous, then u is differentiable on $[a, b]$ and $u' = v$.

Proof. Since u_k is differentiable, it is clear that

$$u_k(x) - u_k(a) = \int_a^x u_k'(t)\, dt, \quad x \in [a, b].$$

By assumption, $u_k' \to v$ uniformly on $[a, b]$, so the previous theorem implies that

$$\int_a^x u_k'(t)\, dt \to \int_a^x v(t)\, dt.$$

Since $u_k \to u$ pointwise on $[a, b]$, we must have that $u(x) - u(a) = \displaystyle\int_a^x v(t)\, dt$. We know that v is continuous, so the fundamental theorem of calculus proves that $u' = v$ on $[a, b]$. □

4 Series

Let $u_0, u_1, u_2, u_3, \ldots$ be a sequence of functions $u_k : I \to \mathbf{C}$, where I is some set. As stated earlier, we define the series $S(x) = \displaystyle\sum_{k=0}^{\infty} u_k(x)$ for those x where the limit exist. This is the pointwise limit of the partial sums $S_n(x) = \displaystyle\sum_{k=0}^{n} u_k(x)$. When does the sequence $S_0, S_1, S_2, \ldots$ converge uniformly? And why would we be interested in this? Well, a rather typical question is if the series converge to something continuous, or differentiable. And whether we can take the derivative of a series — or an integral — *termwise*. In other words, when does a series behave like we are used to when working with a power series? Uniform convergence is a tool to obtain many of these properties and one way of proving uniform convergence is the Weierstrass M-test.

Weierstrass M-test

Theorem. Let $I \subset \mathbf{R}$. Suppose that there exists positive constants M_k, $k = 1, 2, \ldots$, such that $|u_k(x)| \leq M_k$ for $x \in I$. If $\sum_{k=1}^{\infty} M_k < \infty$, then $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on I.

Proof. Since $|u_k(x)| \leq M_k$ and $\sum_{k=1}^{\infty} M_k$ is convergent, it is clear that

$$u(x) = \sum_{k=1}^{\infty} u_k(x)$$

exists for every $x \in I$. Now

$$\begin{aligned}\left\| u(x) - \sum_{k=1}^{n} u_k(x) \right\|_{\infty} &= \left\| \sum_{k=1}^{\infty} u_k(x) - \sum_{k=1}^{n} u_k(x) \right\|_{\infty} = \left\| \sum_{k=n+1}^{\infty} u_k(x) \right\|_{\infty} \\ &\leq \sum_{k=n+1}^{\infty} \|u_k(x)\|_{\infty} \leq \sum_{k=n+1}^{\infty} M_k \to 0,\end{aligned}$$

as $k \to \infty$. By definition, this implies that the series is uniformly convergent. $\square$

By considering the sequence of partial sums $S_n(x)$, $n = 0, 1, 2, \ldots$, of a uniformly convergent series $\sum_{k=0}^{\infty} u_k(x)$, we can express some of the results from the preceding sections in a more convenient form for working with function series.

Series and uniform convergence

Suppose that $u(x) = \sum_{k=0}^{\infty} u_k(x)$ is uniformly convergent for $x \in [a, b]$. If $u_0, u_1, u_2, \ldots$ are continuous functions on $[a, b]$, then the following holds.

(i) The series u is a continuous function on $[a, b]$.

(ii) We can exchange the order of summation and integration:

$$\int_c^d u(x)\, dx = \int_c^d \left(\sum_{k=0}^{\infty} u_k(x) \right) dx = \sum_{k=0}^{\infty} \int_c^d u_k(x)\, dx, \quad \text{for } a \leq c < d \leq b.$$

(iii) If in addition u_k' are continuous and $\sum_{k=0}^{\infty} u_k'(x)$ converges uniformly on $[a, b]$, then

$$u'(x) = \frac{d}{dx} \left(\sum_{k=0}^{\infty} u_k(x)\, dx \right) = \sum_{k=0}^{\infty} \frac{d}{dx} u_k(x) = \sum_{k=0}^{\infty} u_k'(x), \quad x \in [a, b].$$

A couple of remarks:

- All of the above also holds for series of the form $\sum_{k=-\infty}^{\infty} u_k(x)$ when using symmetric partial sums $S_n(x) = \sum_{k=-n}^{n} u_k(x)$.
- In particular, we can use these results for Fourier sums if the conditions are met.
- Recall that all of the results above were satisfied for power series. One way of proving this is to prove that a power series is uniformly convergent inside the region of convergence. We'll get back to this when speaking about the Z transform.

As an example, let's consider a rather famous example.

Example

Let $0 < a < 1$ and $ab > 1$. Show that $u(x) = \sum_{k=1}^{\infty} a^k \sin(b^k \pi x)$ is continuous.

Solution. We see that

$$\left|a^k \sin(b^k \pi x)\right| \leq a^k, \quad k = 1, 2, 3, \ldots,$$

since $|\sin(b^k \pi x)| \leq 1$. Since $\sum_{k=1}^{\infty} a^k$ is a geometric series with quotient a and $|a| < 1$, we know that this series is convergent. Thus, by Weierstrass' M-test, it follows that the original series is uniformly convergent and that u is continuous.

Notice that we didn't calculate the exact $\|\cdot\|_\infty$ norm (well we actually did but we never claimed that the bound was the actual maximum). We just estimated with something that is an upper bound. This is typical (and usually enough). This series is especially interesting since it is an example of a function that is continuous, but *nowhere* differentiable (it is usually referred to as Weierstrass' function). The fact that it is not differentiable is not obvious, but it shows that uniform convergence *isn't* enough to ensure that the limit of something differentiable is differentiable.

In fact, the Weierstrass function does not even have one-sided derivatives (finitely) at any point. So this is an example of a continuous function that definitely does not belong to E'.

5 The Dirichlet Kernel

Consider the complex Fourier series of u. Let us write out and exchange the order of summation and integration according to

$$\begin{aligned} S_n(x) &= \sum_{k=-n}^{n} c_k e^{ikx} = \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_T u(t) e^{-ikt}\, dt \right) e^{ikx} = \frac{1}{2\pi} \int_T u(t) \left(\sum_{k=-n}^{n} e^{-ikt} e^{ikx} \right) dt \\ &= \frac{1}{2\pi} \int_T u(t) \left(\sum_{k=-n}^{n} e^{ik(x-t)} \right) dt. \end{aligned}$$

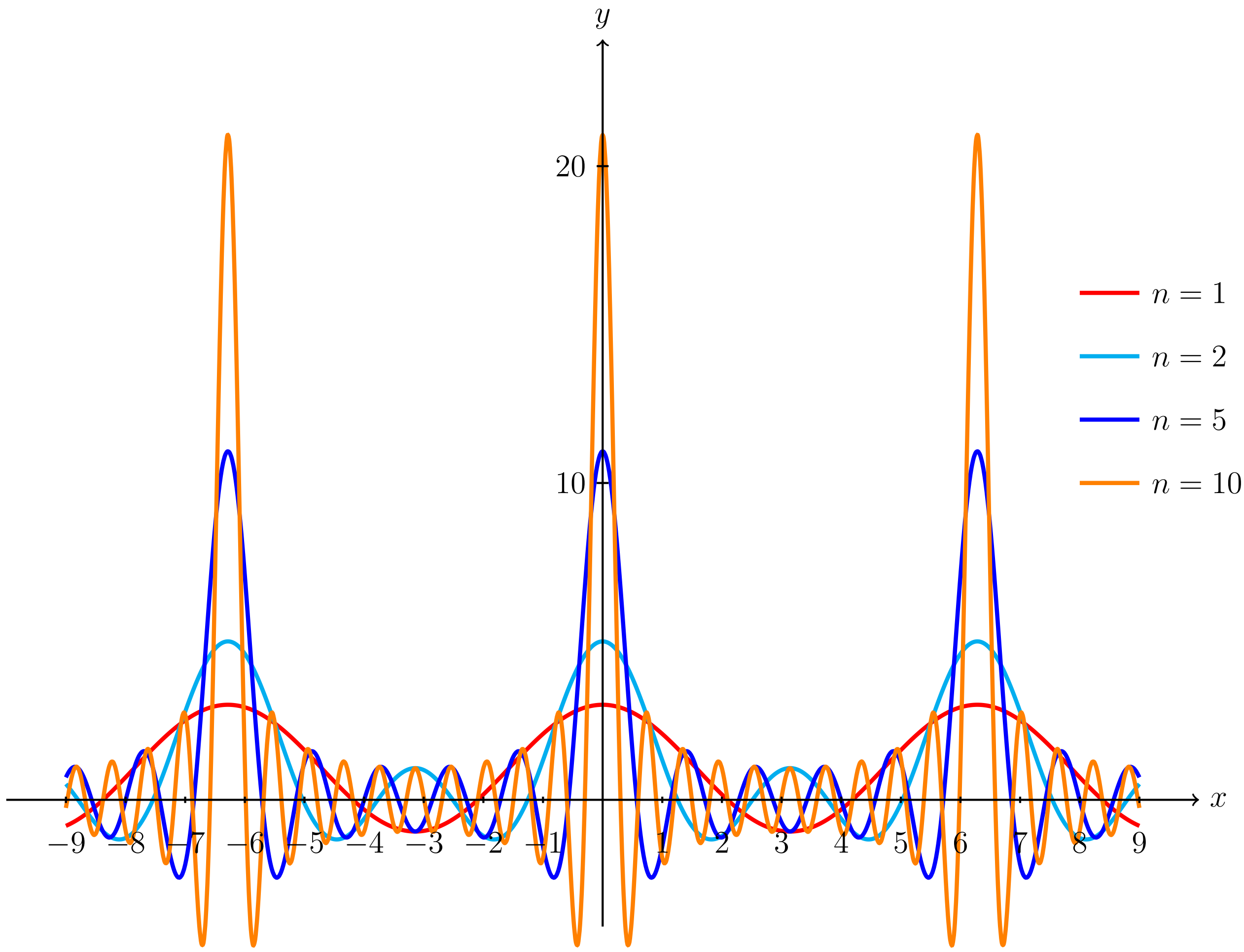
The sum in the parentheses is usually referred to as the Dirichlet kernel.

The Dirichlet kernel

Definition. We define the **Dirichlet kernel** by

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx}, \; x \in \mathbf{R}, \; n = 1, 2, 3, \ldots$$

Directly from the definition, we see that D_n is a real-valued 2π-periodic function. We can plot a couple of instances of D_n.



The greater the value of n, the more D_n oscillates and the peak at multiples of 2π seems to get higher and higher. Moreover, we can now more compactly express the partial sum $S_n u(x)$ as

$$S_n u(x) = \frac{1}{2\pi} \int_T u(t) D_n(x-t)\, dt = \frac{1}{2\pi} \int_T u(s+x) D_n(-s)\, ds = \frac{1}{2\pi} \int_T u(s+x) D_n(s)\, ds,$$

so the partial sums of the Fourier series is given by a *convolution* of u with the Dirichlet kernel (we will get back to convolutions later on). In the first equality, we changed variables $(t - x = s)$ and used the fact that u and D_n are periodic so that we can use the same domain of integration and also that D_n is an even function. The reason for this representation of the partial sums will become clear below.

Let us collect some properties of the Dirichlet kernel.

Theorem.

(i) $D_n(2k\pi) = 2n+1$, $k \in \mathbf{Z}$.

(ii) $D_n(x) = \dfrac{\sin((2n+1)x/2)}{\sin(x/2)}$, $x \neq 2k\pi,\ k \in \mathbf{Z}$.

(iii) $\displaystyle\int_T D_n(x)\,dx = 2\pi$.

Proof.

(i) Since $e^{i2k\pi} = 1$ for $k \in \mathbf{Z}$, it is clear that $D_n(2k\pi) = 2n+1$ since there are $2n+1$ terms in the sum $D_n(x)$.

(ii) For $x \neq 2k\pi$, we observe that $D_n(x)$ is a geometric sum with quotient $e^{ix} \neq 1$, first term e^{-inx} and $2n+1$ terms, so

$$\begin{aligned} D_n(x) &= e^{-inx} \cdot \frac{e^{i(2n+1)x} - 1}{e^{ix} - 1} = e^{-i(n+1/2)x} \cdot \frac{e^{i(2n+1)x} - 1}{e^{ix/2} - e^{-ix/2}} = \frac{e^{i(n+1/2)x} - e^{-i(n+1/2)x}}{e^{ix/2} - e^{-ix/2}} \\ &= \frac{\sin((n+1/2)x)}{\sin(x/2)}. \end{aligned}$$

(iii) We see that

$$\begin{aligned} \int_{-\pi}^{\pi} D_n(x)\,dx &= \int_{-\pi}^{\pi} \left(\sum_{k=-n}^{n} e^{ikx} \right) dx = \int_{-\pi}^{\pi} \left(1 + \sum_{k=1}^{n} \left(e^{ikx} + e^{-ikx} \right) \right) dx \\ &= 2\pi + \sum_{k=1}^{n} 2 \int_{-\pi}^{\pi} \cos kx\,dx = 2\pi, \end{aligned}$$

since all the integrals in the sum are equal to zero. □

6 Pointwise Convergence

We're now going to show one of the main results in this course. So far, we've only calculated the Fourier series for functions without worrying so much (at least explicitly) about convergence. The plots we've seen seem to indicate that the partial sums resemble the function we have found the Fourier series for, but there's been some strange behavior here and there. We have also seen examples where the convergence has been conditional (so not absolute), so even the question if the series converges to *something* has not been clear. So under what conditions can we expect convergence and when will we recover the function we started with?

We now have the tools (this is the main reason for introducing the Dirichlet kernel) to prove that for a function in the space E' (so left- and righthand derivatives exist), the Fourier series actually converges to something that involves the function.

Pointwise convergence (Dirichlet's theorem)

Theorem. Let $u \in E'$. Then

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} \to \frac{u(x^+) + u(x^-)}{2}, \quad x \in [-\pi, \pi].$$

In other words, the Fourier series of u converges pointwise to $\dfrac{u(x^+) + u(x^-)}{2}$ for $x \in [-\pi, \pi]$. In particular, if u also is continuous at x, then $\lim_{n\to\infty} S_n(x) = u(x)$.

Notice the following.

(i) It is sufficient for $u \in E$ (not E') to have left- and righthand derivatives at a specific point x for
$$\lim_{n\to\infty} S_n(x) = \frac{u(x^+) + u(x^-)}{2}$$
to hold at the point x. The condition that $u \in E'$ ensures that this is true for all x.

(ii) The number $(u(\pi^+)+u(\pi^-))/2$ is defined since u is 2π-periodic so that $u(\pi^+) = u((-\pi)^+)$ (the righthand limit at π must be equal to the righthand limit at $-\pi$) and similarly for $u((-\pi)^-)$.

Proof. Let $x \in [-\pi, \pi]$ be fixed (meaning that we won't change the value). We will prove that
$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{\pi} u(x+t) D_n(t)\, dt = \frac{u(x^+)}{2}. \tag{1}$$
A completely analogous argument would show that
$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{0} u(x+t) D_n(t)\, dt = \frac{u(x^-)}{2}$$
and these two limits taken together proves the statement in the theorem.

First we note that
$$\frac{1}{2\pi} \int_0^{\pi} u(x+t) D_n(t)\, dt - \frac{u(x^+)}{2} = \frac{1}{2\pi} \int_0^{\pi} \big(u(x+t) - u(x^+)\big) D_n(t)\, dt$$
since $D_n(t)$ is an even function so $\dfrac{1}{2\pi} \displaystyle\int_0^{\pi} D_n(t)\, dt = \dfrac{1}{2}$ (see the theorem about the Dirichlet kernel above). The same theorem also provides the identity $D_n(x) = \dfrac{\sin((2n+1)x/2)}{\sin(x/2)}$, so
$$\begin{aligned}\big(u(x+t) - u(x^+)\big) D_n(t) &= \big(u(x+t) - u(x^+)\big) \frac{\sin((2n+1)t/2)}{\sin(t/2)} \\ &= \frac{u(x+t) - u(x^+)}{t} \cdot \frac{t}{\sin(t/2)} \cdot \sin(nt + t/2).\end{aligned}$$
Since $u \in E'$, we know that the righthand derivative of u at x exists, so
$$\frac{u(x+t) - u(x^+)}{t} \cdot \frac{t}{\sin(t/2)} \to 2D^+u(x). \tag{2}$$

This means that the expression in the left-hand side of (2) is bounded on $[0, \pi]$ (since it is quite nice outside of the origin). Hence it also belongs to $L^2(0, \pi)$ and E since u is piecewise continuous. Letting

$$v(t) = \begin{cases} \dfrac{u(x+t) - u(x^+)}{t} \cdot \dfrac{t}{\sin(t/2)}, & 0 \le t \le \pi, \\ 0, & -\pi < t < 0, \end{cases}$$

it is clear that $v \in E \subset L^2(-\pi, \pi)$. By the Riemann-Lebesgue lemma, it now follows that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{\pi} \big(u(x+t) - u(x^+)\big) D_n(t)\, dt = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} v(t) \sin\left((n + 1/2)t\right)\, dt = 0,$$

which proves that (1) holds. □

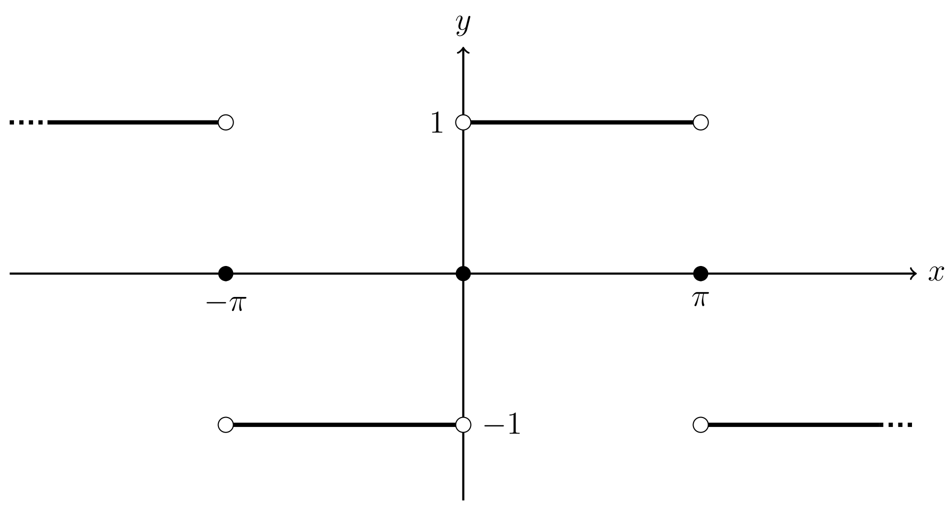
Example

Find the Fourier series for the sign-function $\operatorname{sgn}(x) = -1$ if $x < 0$, $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(x) = 1$ if $x > 0$. Prove when and to what it converges to.

Solution. Let's do this in a backwards manner, just to highlight the point that you don't *need* to know what the Fourier coefficients are to answer the question of convergence! Isn't mathematics just... beautiful.

For $-\pi < x < 0$ and $0 < x < \pi$, u is continuously differentiable so the Fourier series converges to $u(x)$. For $x = 0$, both right- and lefthand derivative exists (both are zero) so the Fourier series converges to $(u(0^-) + u(0^+))/2 = (-1 + 1)/2 = 0$. This happens to be equal to $\operatorname{sgn}(0)$, but this is more of a coincidence. Indeed, we could redefine $\operatorname{sgn}(0) = A$ for any number A we would like and the Fourier series would still converge to 0. For the endpoints, the right- and lefthand derivatives exist (respectively) so the Fourier series converges to $(u(-\pi) + u(\pi))/2 = 0$. Note the analogous situation as that which occurs at $x = 0$. Then the behavior is repeated periodically.

We can now draw the Fourier series since we have analyzed in detail what the series converges to at every point. The type of figure below is what you are supposed to draw when asked to draw the graph of a Fourier series. Do not try to plot the truncated sum.



Again, note that all of this was possible *without* calculating the actual coefficients!

However, we were also asked to find the Fourier series so lets go. The Fourier coefficients can be calculated as follows:

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{sgn}(x)\,dx = 0$$

since sgn is odd, and for $k \neq 0$,

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{sgn}(x)e^{-ikx}\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -e^{-ikx}\,dx + \int_0^{\pi} e^{-ikx}\,dx\right) \\ &= \frac{1}{2\pi}\left(\left[\frac{e^{-ikx}}{ik}\right]_{-\pi}^{0} + \left[-\frac{e^{-ikx}}{ik}\right]_0^{\pi}\right) \\ &= \frac{1}{2\pi}\left(\frac{1}{ik} - \frac{(-1)^k}{ik} - \frac{(-1)^k}{ik} + \frac{1}{ik}\right) = \frac{2(1-(-1)^k)}{2\pi\cdot ik} \\ &= i\,\frac{(-1)^k - 1}{\pi k}. \end{aligned}$$

Hence

$$u(x) \sim \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} i\,\frac{(-1)^k - 1}{\pi k} e^{ikx}.$$

Example

Recall that if $u(x) = x^2$ for $-\pi < x < \pi$, then $u(x) \sim \frac{\pi^2}{3} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{2(-1)^k}{k^2} e^{ikx}$. Use this to evaluate the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$.

Solution. Since x^2 is continuously differentiable on $]-\pi,\pi[$, continuous on $[-\pi,\pi]$ with right- and lefthand derivatives at the endpoints (respectively), and $(-\pi)^2 = \pi^2$, it is clear that the Fourier series of $u(x)$ converges to $u(x)$ for any $x \in [-\pi,\pi]$. Especially this holds for $x = 0$. Therefore

$$0 = u(0) = \frac{\pi^2}{3} + \sum_{k=-\infty,\,k\neq 0}^{\infty} \frac{2(-1)^k}{k^2} e^{ik\cdot 0} = \frac{\pi^2}{3} + 2\sum_{k=1}^{\infty} \frac{2(-1)^k}{k^2} = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2},$$

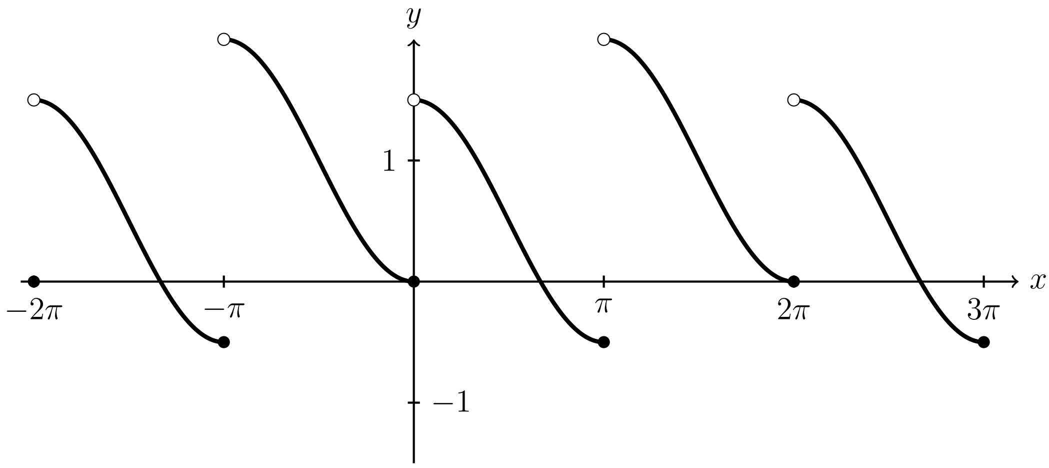
so $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}$.

Could you use the stuff from the previous example to calculate $\sum_{k=1}^{\infty} \frac{1}{k^2}$?

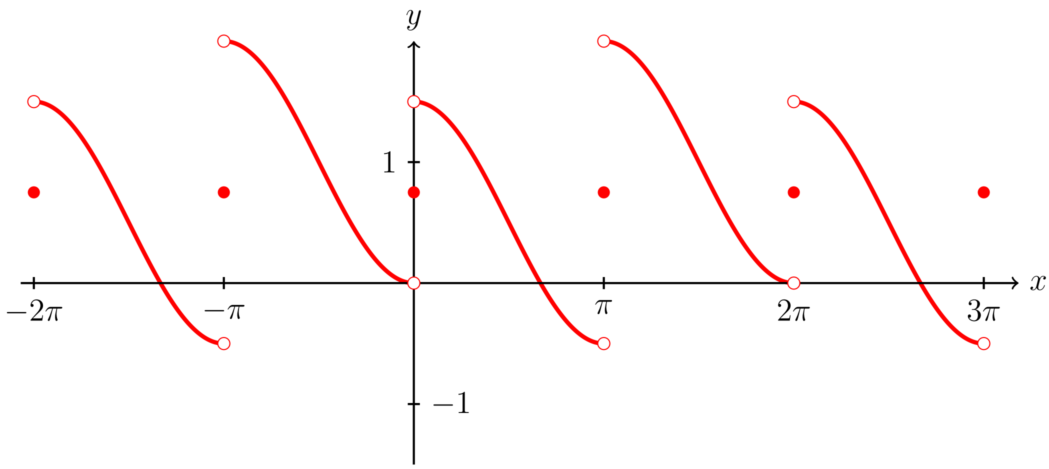
Example

Let $u(x) = \dfrac{1}{2} + \cos x$ for $0 < x \le \pi$, $u(0) = 0$, and $u(x) = 1 - \cos x$ for $-\pi < x < 0$. Show that the Fourier series for the periodic extension of u converges and find the limit of the Fourier series. Where is it equal to u? What's the value of the Fourier series at π? At 2π? Can the convergence be uniform?

Solution. Note again that the we do *not* need to find the Fourier series to answer this question. The function is piecewise continuous and has right- and left-hand derivatives at all points. Hence, by Dirichlet's theorem above, we know that the Fourier series converges to $(u(x^+) + u(x^-))/2$ at all points. So the Fourier series exists (since $u \in E$) and is convergent. As to what the limit $S(x)$ of the Fourier series actually is, let's first draw a graph of the function u (being very careful at the exception points).



So, again, we know that $S(x) = \dfrac{u(x^+) + u(x^-)}{2}$ at every $x \in \mathbf{R}$ since $u \in E$ has right- and lefthand derivatives at every point (is this clear?). For points of continuity of u, that means that $S(x) = u(x)$. For the "jump" points, we take the mean value. This produces the graph below.



What you shouldn't miss here is the fact that it's completely irrelevant what value the function u takes at a single point. It's only the limits of the function that has any effect on the limit of the Fourier series.

From this graph we immediately find that

$$S(\pi) = \frac{u(\pi^-) + u(\pi^+)}{2} = \frac{-1/2 + 2}{2} = \frac{3}{4}$$

and that

$$S(2\pi) = \frac{u(2\pi^-) + u(2\pi^+)}{2} = \frac{0 + 3/2}{2} = \frac{3}{4}.$$

Note in particular that we get the same value, but for two completely different reasons. This is a coincidence (well.. the reason is the symmetry of the function u).

The convergence of the partial sums $S_n(x)$ can not be uniform on any set that includes a point $x = k\pi$ for some integer k. The reason for this is that $S(x)$ is discontinuous at such points, whereas the partial sums $S_n(x)$ (being trigonometric polynomials) are continuous functions on $\mathbf{R}$. Having the limiting function $S(x)$ being discontinuous would violate the convergence being uniform.

Nice conclusion, not a single integral or series in sight!