Lecture 4: Stronger Types of Convergence

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"I did nothing! The pavement was his enemy."
—Julius Benedict

1 Absolute Convergence

So we have obtained sufficient conditions for the Fourier series to converge to the mean value of the left- and righthand limits of the function. Note though that we have said nothing about necessary conditions (and this is not really something we will be able to cover in this course). So let's look in the other direction instead: when can we obtain a stronger type of convergence?

Suppose that $\{c_k\}_{k\in\mathbf{Z}} \in l^1(\mathbf{Z})$, meaning that the series $\displaystyle\sum_{k=-\infty}^{\infty} |c_k| < \infty$, which implies that the Fourier series $\displaystyle\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ converges absolutely. This in turn implies that we have uniform convergence. Let's formulate a theorem.

Theorem. Suppose that $\displaystyle\sum_{k=-\infty}^{\infty} |c_k| < \infty$. Then $\displaystyle\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ converges uniformly.

Proof. Note that

$$S(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

converges for every $x \in \mathbf{R}$ since

$$|S(x) - S_n(x)| = \left| \sum_{|k|>n} c_k e^{ikx} \right| \leq \sum_{|k|>n} |c_k| \to 0,$$

as $n \to \infty$. Note also that the last series is independent of x, which implies uniform convergence:

$$\sup_x |S(x) - S_n(x)| \leq \sum_{|k|>n} |c_k| \to 0,$$

as $n \to \infty$. $\square$

2 A Case Study: $u(x) = x$

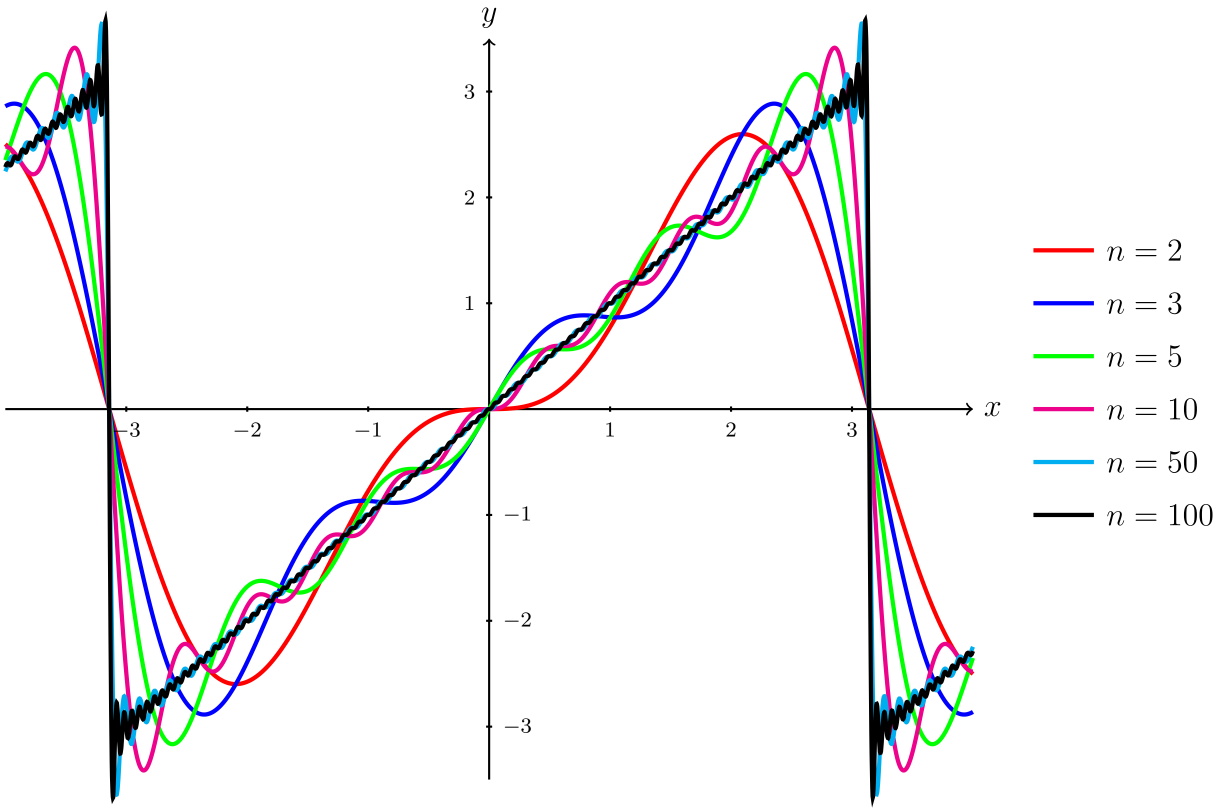
Consider the case when $u(x) = x$ for $-\pi < x < \pi$ (and periodically extended to $\mathbf{R}$). You've seen this before, but let's find the Fourier coefficients. Clearly $c_0 = 0$ (the function is odd) and for $k \neq 0$, integration by parts yields

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-ikx}\,dx = \frac{1}{2\pi}\left(\left[\frac{xe^{-ikx}}{-ik}\right]_{-\pi}^{\pi} + \frac{1}{ik}\int_{-\pi}^{\pi} e^{-ikx}\,dx\right) = \frac{1}{2\pi}\left(\frac{2\pi(-1)^k}{-ik} + 0\right)$$
$$= \frac{(-1)^{k+1}}{ik}.$$

Since u is continuously differentiable for $-\pi < x < \pi$, we know from the previous lecture that

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = x, \quad -\pi < x < \pi.$$

We also know that the Fourier series converges to 0 at $x = \pm\pi$. Since the limit function is discontinuous at $\pm\pi$, the convergence can *not* be uniform on $[-\pi, \pi]$, but there's a possibility that the convergence *is* uniform for $a < x < b$ with $-\pi < a < b < \pi$.



Let's show the following fact.

Theorem. For $0 < b < \pi$, the Fourier series for $u(x) = x$, $-\pi < x < \pi$, converges uniformly on $[-b, b]$.

Proof. Suppose that $m, n \in \mathbf{N}$ and that $m > n$. Let

$$S_l(x) = \sum_{k=-l}^{l} c_k e^{ikx}, \quad l = 1, 2, 3, \dots,$$

be the partial sums for the Fourier series of $u(x) = x$, $-\pi < x < \pi$. We will show that

$$\sup_{x \in [-b,b]} |S_m(x) - S_n(x)| \to 0,$$

as $m, n \to \infty$. This is sufficient for obtaining uniform convergence (and is known as Cauchy's criterion for uniform convergence). First we note that

$$\begin{aligned} S_m(x) - S_n(x) &= \sum_{k=-m}^{m} c_k e^{ikx} - \sum_{k=-n}^{n} c_k e^{ikx} = \sum_{k=-m}^{-(n+1)} c_k e^{ikx} - \sum_{k=n+1}^{m} c_k e^{ikx} \\ &= \sum_{k=n+1}^{m} \left(\frac{(-1)^{k+1}}{ik} e^{ikx} + \frac{(-1)^{-k+1}}{-ik} e^{-ikx} \right) = \sum_{k=n+1}^{m} \frac{(-1)^{k+1}}{ik} \left(e^{ikx} - e^{-ikx} \right). \end{aligned}$$

We need to exploit the fact that the terms are both positive and negative to show that this is small. For this purpose, we observe that $\delta = \cos(b/2) > \cos(\pi/2) = 0$. If we were to multiply a term in the series by $\cos(x/2)$, we would obtain

$$\begin{aligned} \cos\left(\frac{x}{2}\right) \frac{(-1)^{k+1}}{ik} \left(e^{ikx} - e^{-ikx} \right) &= \frac{(-1)^{k+1}}{i2k} \left(e^{ix/2} + e^{-ix/2} \right) \left(e^{ikx} - e^{-ikx} \right) \\ &= \frac{(-1)^{k+1}}{i2k} \left(e^{i(k+1/2)x} + e^{i(k-1/2)x} - e^{-i(k-1/2)x} - e^{-i(k+1/2)x} \right) \\ &= \frac{(-1)^{k+1}}{k} \left(\sin\left(k + \frac{1}{2}\right) x + \sin\left(k - \frac{1}{2}\right) x \right). \end{aligned}$$

Examining the sum of these terms more closely, we find that

$$\sum_{k=n+1}^{m} \frac{(-1)^{k+1}}{k} \left(\sin\left(k + \frac{1}{2}\right) x + \sin\left(k - \frac{1}{2}\right) x \right)$$

is equal to

$$\begin{aligned} (-1)^{n+1} \Bigg(& \frac{\sin\left(n + \frac{3}{2}\right) x + \sin\left(n + \frac{1}{2}\right) x}{n+1} - \frac{\sin\left(n + \frac{5}{2}\right) x + \sin\left(n + \frac{3}{2}\right) x}{n+2} \\ & + \frac{\sin\left(n + \frac{7}{2}\right) x + \sin\left(n + \frac{5}{2}\right) x}{n+3} - \frac{\sin\left(n + \frac{9}{2}\right) x + \sin\left(n + \frac{7}{2}\right) x}{n+4} \\ & + \cdots \pm \frac{\sin\left(m - \frac{1}{2}\right) x + \sin\left(m - \frac{3}{2}\right) x}{m-1} \mp \frac{\sin\left(m + \frac{1}{2}\right) x + \sin\left(m - \frac{1}{2}\right) x}{m} \Bigg). \end{aligned}$$

We can rearrange the terms in the parenthesis as

$$\frac{\sin\left(n + \frac{1}{2}\right) x}{n+1} + (-1)^{n+1} \sum_{k=n+1}^{m-1} (-1)^k \left(\frac{1}{k} - \frac{1}{k+1} \right) \sin\left(k + \frac{1}{2}\right) x + (-1)^m \frac{\sin\left(m + \frac{1}{2}\right) x}{m}.$$

Using the fact that $|\sin t| \leq 1$ for $t \in \mathbf{R}$, we now obtain that

$$|S_m(x) - S_n(x)| \leq \delta^{-1} \left(\frac{1}{n+1} + \sum_{k=n+1}^{m-1} \left| \frac{1}{k} - \frac{1}{k+1} \right| + \frac{1}{m} \right)$$
$$\leq \delta^{-1} \left(\frac{1}{n+1} + \sum_{k=n+1}^{m-1} \frac{1}{k(k+1)} + \frac{1}{m} \right) \leq \delta^{-1} \left(\frac{2}{n} + \sum_{k=n+1}^{\infty} \frac{1}{k^2} \right) \to 0,$$

as $n \to \infty$ since the series of $1/k^2$ is convergent and $2/n \to 0$. □

So that was awesome (or absolutely positively horrifying). We'll need the Fourier expansion of x later on in this lecture.

3 Uniform Convergence

So as seen from the previous case study, proving uniform convergence directly can be rather cumbersome. And obviously, demanding that we have absolute convergence is rather restrictive. We would prefer less draconian requirements that are easier to verify. Too much to ask for? Not really!

Theorem. Suppose that u is continuous on $[-\pi, \pi]$, that $u(-\pi) = u(\pi)$ and that $u' \in E$. Then the Fourier series of u converges uniformly to u on $[-\pi, \pi]$.

Before proving this theorem, let's make a comment on the condition that $u' \in E$. How should this be interpreted? In this course, this expression does *not* mean that u is differentiable everywhere. It only means that u' exists except for possibly a finite number of exception points in each finite interval. Between these exception points, u' is continuous and one-sided limits of u' exists at the exception points (these are just the requirements of the space E). This is why we need the additional assumption that u is continuous (which is not necessarily true if we only know that $u' \in E$ with this interpretation). Note that $u' \in E$ and u being continuous is sufficient for handling typical integrals involving u'. Huh? Suppose that c is the only exception point of u' in $[a, b]$. Then, for example,

$$\int_a^b u'(x)\,dx = \int_a^c u'(x)\,dx + \int_c^b u'(x)\,dx = u(c^-) - u(a) + u(b) - u(c^+) = u(b) - u(a)$$

since u is continuous so that $u(c^+) = u(c^-)$. So the answer is the same as what we would get if u' existed everywhere. We're now ready for the proof of the theorem!

Proof. Since $u' \in E$, we know that u' has a Fourier series

$$u'(x) \sim \sum_{k=-\infty}^{\infty} d_k e^{ikx},$$

where

$$d_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} u'(t) e^{-ikt}\,dt, \quad k \in \mathbf{Z}.$$

Furthermore, the fact that $u(-\pi) = u(\pi)$ implies that $d_0 = 0$:

$$d_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u'(t)\,dt = \frac{u(\pi)-u(-\pi)}{2\pi} = 0.$$

Now, since $u' \in E$, it is clear that $u \in E'$, which implies that u has a convergent Fourier series

$$u(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx},$$

where the equality follows from the fact that u is continuous. Here

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)e^{-ikt}\,dt, \quad k \in \mathbf{Z}.$$

How are c_k and d_k related? Assuming that $k \neq 0$, we have

$$\begin{aligned} d_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u'(t)e^{-ikt}\,dt = \text{/ I.B.P. /} = \frac{1}{2\pi}\left(\left[u(t)e^{-ikt}\right]_{-\pi}^{\pi} + ik\int_{-\pi}^{\pi} u(t)e^{-ikt}\,dt\right) \\ &= ikc_k \end{aligned}$$

since $u(\pi)e^{-ik\pi} = u(-\pi)e^{ik\pi}$ for $k \in \mathbf{Z}$. Now, Bessel's inequality shows that

$$\sum_{k=-\infty}^{\infty} |d_k|^2 \leq \|u'\|^2_{L^2(-\pi,\pi)} < \infty,$$

and since $d_k = ikc_k$, this implies that

$$\sum_{k=-\infty}^{\infty} k^2|c_k|^2 \leq \|u'\|^2_{L^2(-\pi,\pi)} < \infty.$$

Note that we could have used Parseval's identity in the place of Bessel's inequality, but we haven't shown why this holds yet. Now, by the Cauchy-Schwarz inequality,

$$\sum_{k=1}^{\infty} |c_k| = \sum_{k=1}^{\infty}\left|\frac{1}{k}\cdot kc_k\right| \leq \left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{1/2}\left(\sum_{k=1}^{\infty}k^2|c_k|^2\right)^{1/2} < \infty,$$

and similarly for $k < 0$. By Weierstrass' M-test, it now follows that $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ is uniformly convergent since $|c_k e^{ikx}| \leq |c_k|$ due to the fact that $|e^{ikx}| = 1$. □

Smoothness and convergence

The proof of the previous theorem exploits the fact that u is quite smooth (meaning that u' exists in some sense) to obtain that the Fourier coefficients of u tend to zero faster than if u was not smooth. This is something important when it comes to Fourier analysis: a smoother function provides better convergence. What could we do if u is twice differentiable?

We can also "localize" the previous theorem to show that we actually have uniform convergence on any interval $[a,b] \subset [-\pi,\pi]$ such that u is continuous with a piecewise continuous derivative.

Theorem. Suppose that u is continuous on $[a, b] \subset]-\pi, \pi[$ and that $u, u' \in E[-\pi, \pi]$. Then the Fourier series of u converges uniformly to u on $[a, b]$.

Proof. We will use that fact that the Fourier series for $v(x) = x$ converges uniformly on every interval $[-c, c] \subset]-\pi, \pi[$ to modify $u(x)$ so that we can apply the previous (global) uniform convergence result. To this end, let $v(x) = x$ for $-\pi < x < \pi$ and $v(\pm\pi) = 0$. Moreover, let

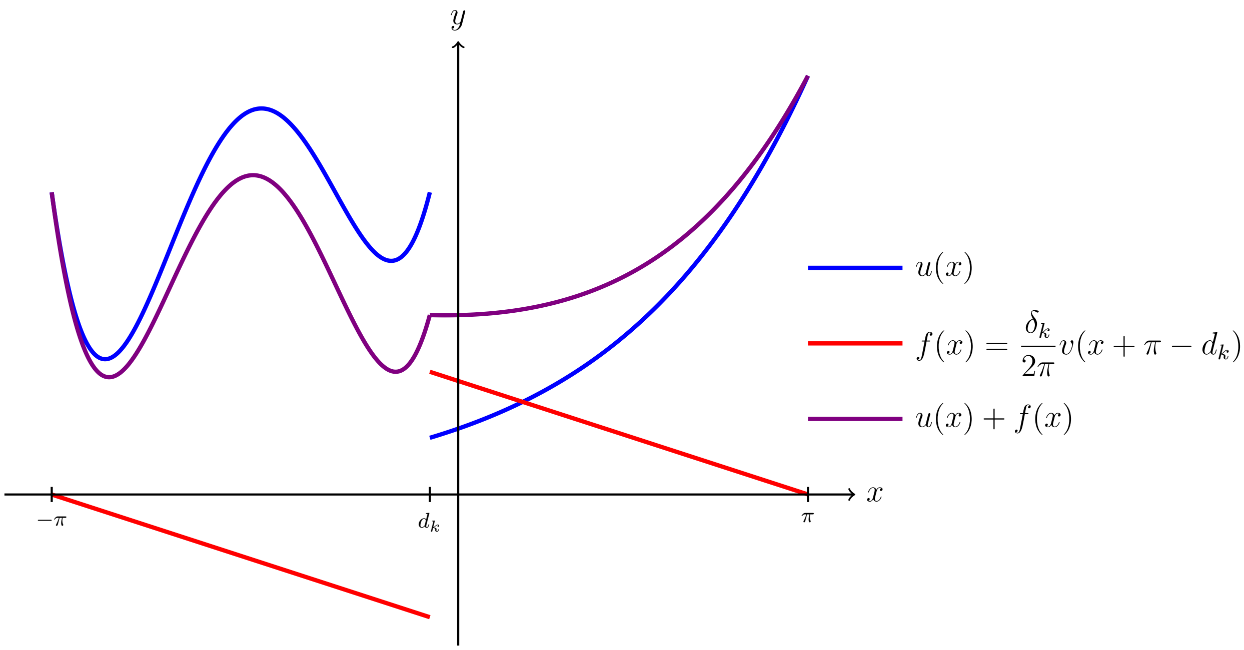
$$-\pi < d_1 < d_2 < \cdots < d_n \leq \pi,$$

where d_k are the points of discontinuity of u (where the function "jumps"). We can also redefine $u(\pm\pi)$ so that these values are equal (and thereby possible adding a new point d_k). This will not affect the result since $[a, b] \subset] - \pi, \pi[$. Furthermore, define

$$\delta_k = u(d_k^+) - u(d_k^-), \quad k = 1, 2, \ldots, n.$$

To obtain a function continuous on $[-\pi, \pi]$, we consider the following construction:

$$w(x) = u(x) + \sum_{k=1}^{n} \frac{\delta_k}{2\pi} v(x + \pi - d_k), \quad x \in [-\pi, \pi].$$



Since $v(x)$ has a jump at $x = \pm\pi + 2\pi k$ of the size 2π (jumps from π to $-\pi$), it is clear that

$$\begin{aligned} w(d_m^+) - w(d_m^-) &= \delta_m + \frac{\delta_m}{2\pi} v(d_m^+ + \pi - d_m) - \frac{\delta_m}{2\pi} v(d_m^- + \pi - d_m) \\ &= \delta_m \left(1 + \frac{v(d_m^+ + \pi - d_m) - v(d_m^- + \pi - d_m)}{2\pi}\right) = \delta_m \left(1 + \frac{-\pi - \pi}{2\pi}\right) = 0 \end{aligned}$$

for $m = 1, 2, \ldots, n$. If $x \neq d_k$ for every $k = 1, 2, \ldots, n$, then w is continuous since both u and $x \mapsto v(x + \pi + d_k)$ are continuous at x. After possible redefinition at the points $\{d_k\}$, we have shown that w is continuous on $[-\pi, \pi]$ and that $w(\pi) = w(-\pi)$.

The previous theorem then proves that the Fourier series of w converges uniformly on $[-\pi,\pi]$. Moreover, we know that $v(x)=x$ has a Fourier series that converges uniformly on $[-c,c]$ for any $0<c<\pi$. This implies that the Fourier series of $\frac{\delta_k}{2\pi}v(x+\pi-d_k)$ converges uniformly on any interval $[a,b]\subset]-\pi,\pi[$ that does not contain d_k. By assumption, u is continuous on $[a,b]$ so there are no jump points in $[a,b]$. Since the Fourier series of both $w(x)$ and $\sum_{k=1}^{n}\frac{\delta_k}{2\pi}v(x+\pi-d_k)$ converges uniformly on $[a,b]$, this must also hold for the Fourier series of $u(x)$ on $[a,b]$. □

4 Periodic Solutions to Differential Equations

Remember the most fun part in TATA42? Yeah, me too! Suppose that we wish to find a periodic solution to a differential equation, could we make an ansatz and find a solution as its Fourier series? Hypothetically yes, but the theory is a bit more difficult and subtle than the corresponding case with a power series approach. Let's consider an example.

Notice first that if y is differentiable and periodic, then y' is also periodic with the same period. Indeed,

$$y'(x+T)=\lim_{h\to 0}\frac{y(x+T+h)-y(x+T)}{h}=\lim_{h\to 0}\frac{y(x+h)-y(x)}{h}=y'(x).$$

Example

Find all 2π-periodic solutions to $y''(x)+2y(x+\pi)=\cos x$.

Solution. So, the plan is to assume that $y(x)=\sum_{k=-\infty}^{\infty}c_ke^{ikx}$ and plug this into the equation and identify the coefficients c_k (just as in TATA42). Note though, that we expressed that $y(x)$ was *equal* to its Fourier series above. This is not clear without motivation, so here goes.

Reformulating the equation, we see that $y''=\cos x-2y(x+\pi)$. Since we're seeking a function that's at least differentiable, this means that y must be continuous. Hence y'' is also continuous. Why? Well,

$$y''=-2y(x+\pi)+\cos x \tag{1}$$

so since both y and $\cos x$ are continuous, this must mean that y'' is also continuous. This means that $y\in C^2(\mathbf{R})$. Therefore, the right-hand side of (1) is obviously twice differentiable and so y''' must be continuous. Hence $y\in C^3(\mathbf{R})$. This is sufficient for letting

$$\begin{aligned} y(x)&=\sum_{k=-\infty}^{\infty}c_ke^{ikx},\\ y'(x)&=\sum_{k=-\infty}^{\infty}ikc_ke^{ikx},\\ y''(x)&=\sum_{k=-\infty}^{\infty}-k^2c_ke^{ikx},\end{aligned}$$

something that is clear from Dirichlet's theorem (if $f\in E'$ is continuous then the Fourier series converges to f) and the fact that we can form the termwise derivative (twice) due to uniform convergence (periodicity and $y \in C^3$ is sufficient for this due to the previous theorem). Furthermore,

$$y(x+\pi) = \sum_{k=-\infty}^{\infty} c_k e^{ik(x+\pi)} = \sum_{k=-\infty}^{\infty} c_k e^{ik\pi} e^{ikx} = \sum_{k=-\infty}^{\infty} c_k (-1)^k e^{ikx}.$$

Therefore, we must have

$$y''(x) + 2y(x+\pi) = \cos x \quad \Leftrightarrow \quad \sum_{k=-\infty}^{\infty} (-k^2 + 2(-1)^k) c_k e^{ikx} = \cos x = \frac{1}{2} e^{ix} + \frac{1}{2} e^{-ix}.$$

By uniqueness (we're looking for continuous functions), it then follows that

$$\begin{aligned} c_k(-k^2 + 2(-1)^k) &= 1/2, \quad k = \pm 1, \\ c_k(-k^2 + 2(-1)^k) &= 0, \quad k \neq \pm 1. \end{aligned}$$

So $c_{-1} = c_1 = -1/6$. For $k \neq \pm 1$, we must have $c_k = 0$ or $-k^2 + 2(-1)^k = 0$. Clearly

$$-k^2 + 2(-1)^k = 0 \quad \Leftrightarrow \quad k^2 = 2(-1)^k$$

has no solutions in $\mathbf{Z}$ since either $k^2 = -2$ (nothing real) or $k^2 = 2$ (nothing integer valued or even rational). We have now obtained that

$$y(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} = -\frac{1}{6} e^{-ix} - \frac{1}{6} e^{ix} = -\frac{1}{3} \cos x.$$

Example

Show that there's no non-zero 2π-periodic solution to

$$y'(t) + 2y(t + \pi/3) = 0, \quad t \in \mathbf{R}.$$

Solution. Since y needs to be differentiable, we know that y is continuous. The fact that

$$y'(t) = -2y(t + \pi/3)$$

then implies that y' is continuous. Hence y is $C^1(\mathbf{R})$. But then $y' \in C^1(\mathbf{R})$ as well, which means that $y \in C^2(\mathbf{R})$. This continues ad nauseum, so y must be a very smooth function. For our purposes, $y \in C^2(\mathbf{R})$ is sufficient for writing that

$$y(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \quad \text{and} \quad y'(t) = \sum_{k=-\infty}^{\infty} ikc_k e^{ikt}.$$

Why? Well if $y \in C^2(\mathbf{R})$, then both y and y' have uniformly convergent Fourier series and the Fourier series of y' can be found through termwise differentiation of the Fourier series for y. Hence

$$0 = y'(t) + 2y(t + \pi/3) = \sum_{k=-\infty}^{\infty} \left(ik + 2e^{ik\pi/3}\right) c_k e^{ikt}.$$

Hence $ik + 2e^{ik\pi/3} = 0$ or $c_k = 0$. We note that for $k = 0$, we have $i \cdot 0 + 2 \neq 0$ so $c_0 = 0$. If $k = \pm 1$, then

$$i(\pm 1) + 2e^{i(\pm 1)\pi/3} = i(\pm 1) + 2\left(\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right) \neq 0$$

so $c_{\pm 1} = 0$. Similarly,

$$i(\pm 2) + 2e^{i(\pm 2)\pi/3} = i(\pm 2) + 2\left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right) \neq 0$$

so $c_{\pm 2} = 0$ is necessary. For $|k| > 2$, it is impossible to find a solution to $ik + 2e^{ik\pi/3}$ since $|2e^{ik\pi/3}| = 2$. Hence $c_k = 0$ if $|k| > 2$. Therefore, only $y(t) = 0$ is 2π-periodic and solves the equation.

5 Rules for Calculating Fourier Coefficients

Linearity

Theorem. Suppose that $u, v \in E$. Then $\widehat{au + bv}[k] = a\widehat{u}[k] + b\widehat{v}[k]$.

Proof. This follows from the linearity of the integral and the fact that everything is convergent for functions in E. □

We've already seen the following result in the previous sections.

Differentiation

Theorem. Suppose that $u' \in E$ and u is continuous with $u(-\pi) = u(\pi)$. If $u \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$, then $u'(x) \sim \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}$. That is, $\widehat{u'}[k] = ik\widehat{u}[k]$.

Example

Let $u(x) = |x|$ for $-\pi \leq x \leq \pi$. Use the fact that $u'(x) = \operatorname{sgn}(x)$ for $0 < |x| < \pi$ to find the Fourier series of $\operatorname{sgn}(x)$.

Solution. Recall that

$$u(x) = \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^k - 1}{\pi k^2} e^{ikx},$$

and that we have equality since $u \in E'$ and u is continuous. The Fourier coefficients are $c_0 = \pi/2$ and $c_k = \dfrac{(-1)^k - 1}{\pi k^2}$ for $k \neq 0$. Noting that $u'(x) = -1$ if $-\pi < x < 0$ and $u'(x) = 1$ if $0 < x < \pi$, we see that $u'(x) = \operatorname{sgn}(x)$ when $0 < |x| < \pi$. Hence $\operatorname{sgn}(x)$ has the Fourier coefficients $d_0 = 0$ and

$$d_k = ikc_k = ik\frac{(-1)^k - 1}{\pi k^2} = i\frac{(-1)^k - 1}{\pi k}, \quad k \neq 0,$$

and thus the Fourier series

$$\operatorname{sgn}(x) \sim \frac{i}{\pi} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^k - 1}{k}.$$

This is true since $u' \in E$, $u(-\pi) = u(\pi)$ and u is continuous. Observe also that it doesn't matter that $u'(x) \neq \operatorname{sgn}(x)$ at some points. In fact, as long as the set of points where $u'(x) \neq \operatorname{sgn}(x)$ is small enough that it doesn't affect the integration, we'll obtain the same Fourier series. This is true for any Fourier series calculation. However, what the Fourier series converges to might not be the function at these exception points.

Mirroring

Theorem. Suppose that $u \in E$. Then $\widehat{u(-x)}[k] = \widehat{u(x)}[-k]$ for $k \in \mathbf{Z}$.

Proof:

$$\int_{-\pi}^{\pi} u(-t)e^{-ikt}\,dt = /s = -t/ = -\int_{\pi}^{-\pi} u(s)e^{iks}\,ds = \int_{-\pi}^{\pi} u(s)e^{-i(-k)s}\,dt = 2\pi c_{-k}. \qquad \square$$

Conjugation

Theorem. Suppose that $u \in E$. Then $\widehat{\overline{u(x)}}[k] = \overline{\widehat{u}[-k]}$ for $k \in \mathbf{Z}$.

Proof:

$$\int_{-\pi}^{\pi} \overline{u(t)}e^{-ikt}\,dt = \int_{-\pi}^{\pi} \overline{u(t)e^{ikt}}\,dt = \overline{\int_{-\pi}^{\pi} u(t)e^{-i(-k)t}\,dt} = 2\pi\overline{c_{-k}}. \qquad \square$$

Translation

Theorem. Suppose that $u \in E$. Then $\widehat{u(x-y)}[k] = e^{-iky}\widehat{u}[k]$ for $k \in \mathbf{Z}$.

Proof:

$$\int_{-\pi}^{\pi} u(x-y)e^{-ikx}\,dx = /\ t = x - y\ / = \int_{-\pi-y}^{\pi-y} u(t)e^{-ik(t+y)}\,dt = e^{-iky}\int_{-\pi-y}^{\pi-y} u(t)e^{-ikt}\,dt$$

$$= /\ u \text{ is } 2\pi\text{-periodic}\ / = e^{-iky}\int_{-\pi}^{\pi} u(t)e^{-ikt}\,dt = e^{-iky}2\pi c_k. \qquad \square$$

Example

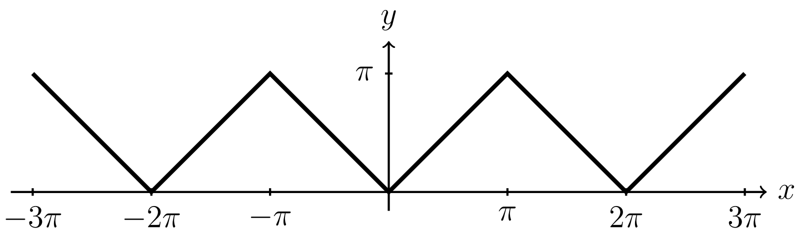
Let $u(x) = |x|$ for $-\pi \leq x \leq \pi$ and extend u periodically. Find the Fourier coefficients for $u(x-1)$. To what does the Fourier series converge? What is the Fourier series for the function $w(x) = |x-1|$, $-\pi \leq x \leq \pi$?

Solution. This is a good example since it shows the dangers of not remembering that we're working with periodic extensions outside the domain $[-\pi, \pi]$.

Recall that

$$u(x) = \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^k - 1}{\pi k^2} e^{ikx}.$$

Drawing the function looks like this.



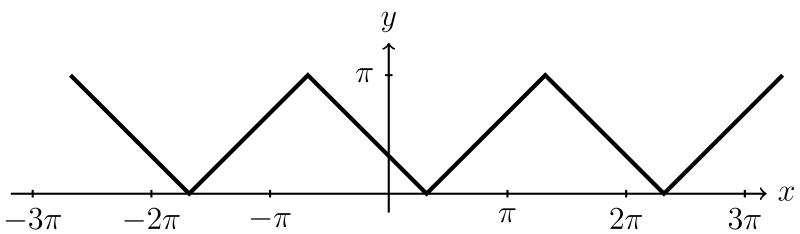
It is a periodic function. Now, finding the Fourier coefficients for $u(x-1)$ is rather easy if we use the "rule" above:

$$\widehat{u(x-1)}[k] = e^{-ik}\widehat{u}[k] = e^{-ik}\frac{(-1)^k - 1}{\pi k^2}, \quad k \neq 0,$$

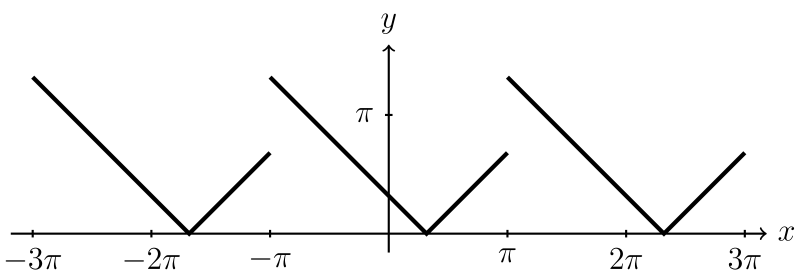
and $\widehat{u(x-1)}[0] = e^{-i\cdot 0}\frac{\pi}{2} = \frac{\pi}{2}$. This means that

$$u(x-1) = \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} e^{-ik}\frac{(-1)^k - 1}{\pi k^2} e^{ikx},$$

again with equality since $u(x-1)$ is a continuous function in E'. Note now though what the graph looks like.



It is a shifted copy of the graph of $u(x)$, which was something that we extended periodically. If we *actually* want to find the Fourier series for $w(x) = |x-1|$, $-\pi \leq x \leq \pi$, we have to do a new calculation and this would look different. Furthermore, the Fourier series will not converge to $|x-1|$ at $\pm\pi$. Drawing what $w(x)$ actually looks like (and extending w periodically) makes this clear.



Doing the calculation, you would find that

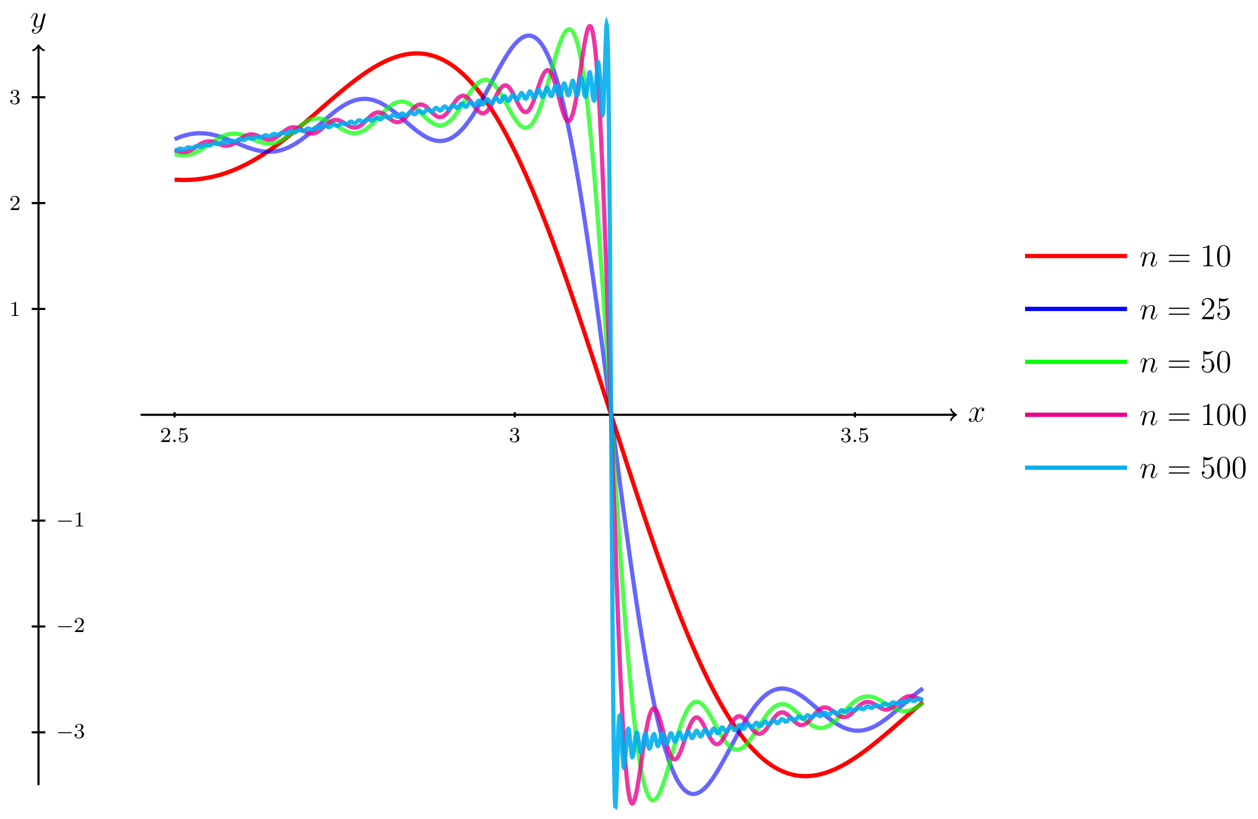
$$w(x) \sim \frac{1+\pi^2}{2\pi} + \frac{1}{\pi} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^k(1-ik) - e^{-ik}}{k^2}.$$

When is $w(x)$ equal to it's Fourier series?

6 Gibbs' Phenomenon

As we saw in the first example of this lecture, the Fourier series of $v(x) = x$ behaves rather odd at the points $x = \pm\pi$. We've seen this squiggly behavior previously as well. For example when looking at the Fourier series for $\operatorname{sgn}(x)$ we saw that this happened at the origin. However, the function $x \mapsto x^2$ did not exhibit the oscillatory stuff. Why? The reason is the continuity. We've seen in this lecture that we have uniform convergence on closed intervals under rather generous conditions that include continuity of the function, and what is common with all the squiggly sums is that the functions have a jump around which the Fourier partial sums oscillate. Let's take a zoomed in look at the partial sums for $v(x) = x$, that is

$$T_m(x) = \sum_{k=1}^{m} \frac{2(-1)^{k+1}}{k} \sin kx.$$



What is very interesting in this picture is that the height of the oscillations (at the extremes) seems to be the same no matter how many terms we use in the partial sum. What changes is that the oscillations gets more squeezed together around the jump point. Now this is just a specific example, but it turns out that this holds for *all* functions with jump discontinuities.

The height of the wobbliness is about 9% of the size of the jump. This is known as Gibbs' phenomenon.

To see why this holds in this case (which we will use to show the general case below), consider the sequence $\{x_m\}$ defined by $x_m = \pi(1 - m^{-1})$. Then

$$\begin{aligned} T_m(x_m) &= \sum_{k=1}^{m} \frac{2(-1)^{k+1}}{k} \sin\left(k\pi\left(1 - \frac{1}{m}\right)\right) \\ &= \sum_{k=1}^{m} \frac{2(-1)^{k+1}}{k} \left(\sin k\pi \cos\left(\frac{\pi}{m}\right) - \cos k\pi \sin\left(\frac{k\pi}{m}\right)\right) \\ &= \sum_{k=1}^{m} \frac{2(-1)^{2k+2}}{k} \sin\left(\frac{k\pi}{m}\right) = \sum_{k=1}^{m} \frac{2}{k} \sin\left(\frac{k\pi}{m}\right) = 2\sum_{k=1}^{m} \frac{\sin\left(\frac{k\pi}{m}\right)}{\frac{k\pi}{m}} \cdot \frac{\pi}{m}. \end{aligned}$$

Next we observe that this is a Riemann sum for the function $x \mapsto \dfrac{\sin x}{x}$ on the interval $[0, \pi]$, so since this function is Riemann integrable, letting $m \to \infty$ yields that

$$\lim_{m\to\infty} T_m(x_m) = 2\int_0^{\pi} \frac{\sin x}{x}\, dx \geq 1.18\pi.$$

Furthermore, certainly $v(x_m) \to \pi$ increasingly as $m \to \infty$ and $v(\pi^+) - v(\pi^-) = -2\pi$, so the size of the jump is 2π. Therefore,

$$\lim_{m\to\infty} \frac{T_m(x_m) - v(x_m)}{2\pi} \approx \frac{1.18\pi - \pi}{2\pi} = 0.09,$$

so for m large enough, we have

$$\frac{T_m(x_m) - v(x_m)}{2\pi} \geq 0.089.$$

This means that there's a sequence $\{x_m\}$ for which the difference between $T_m(x_m)$ and $v(x_m)$ is about 9% of the the size of the jump!

To summarize, we have a sequence $\{x_m\}$ such that $x_m < \pi$ and $x_m \to \pi^-$ increasingly and for which

$$\frac{T_m(x_m) - v(x_m)}{2\pi} \geq 0.089,$$

for m large enough. Completely analogously, there exists a sequence $\{x_m\}$ such that $x_m > \pi$ and $x_m \to \pi^+$ decreasingly and for which

$$\frac{T_m(x_m) - v(x_m)}{2\pi} \leq -0.089$$

for m large enough.

What's even more impressive is that this fact holds for all jumps when dealing with functions from E. The size of the oscillations are always about 9% of the size of the jump. So large jumps (like at the endpoints) cause a lot of "overshooting" where a signal will look weird (and the amplitude will overshoot the "expected" signal). To formalize this, we have the following theorem.

Theorem. Suppose that $u \in E$ is continuous on the interval $[d-\delta, d+\delta]$ except at $x = d$ and suppose that $u' \in E$. Let $\delta_d = u(d^+) - u(d^-)$. Then there exists a sequence $x_m \to d^+$ such that

$$\lim_{m\to\infty} \frac{S_m(x_m) - u(x_m)}{\delta_d} \geq 0.089,$$

where $S_m(x)$ are the partial Fourier sums of u.

Proof. Using the same idea that was used when proving uniform convergence on subintervals, let us define the function w by

$$w(x) = u(x) + \frac{\delta_d}{2\pi} v(x + \pi - d),$$

where $v(x) = x$ for $-\pi < x < \pi$ and $v(\pm\pi) = 0$. Then w is continuous on $I = [d-\delta, d+\delta]$ and since $w, w' \in E$, the Fourier series of w converges uniformly on I. Thus, for any $\epsilon > 0$, we may assume that $\delta > 0$ is small enough such that

$$|W_m(x) - w(x)| < |\delta_d|\epsilon$$

for $m \geq N$ (some $N \in \mathbf{Z}$) and all $x \in I$, where $W_m(x)$ is the partial Fourier series of $w(x)$. Since

$$W_m(x) = S_m(x) + \frac{\delta_d}{2\pi} T_m(x + \pi - d), \quad m = 1, 2, 3, \ldots,$$

it follows that

$$\begin{aligned}\frac{S_m(x) - u(x)}{\delta_d} &= \frac{W_m(x) - \frac{\delta_d}{2\pi} T_m(x+\pi-d) - (w(x) - \frac{\delta_d}{2\pi} v(x+\pi-d))}{\delta_d} \\ &= \frac{W_m(x) - w(x)}{\delta_d} - \frac{T_m(x+\pi-d) - v(x+\pi-d)}{2\pi}.\end{aligned}$$

By the argument above, there is a sequence $x_m > d$ such that

$$\frac{T_m(x+\pi-d) - v(x+\pi-d)}{2\pi} \leq -0.089,$$

from which it is clear that

$$\lim_{m\to\infty} \frac{S_m(x_m) - u(x_m)}{\delta_d} \geq 0.089,$$

since ϵ was arbitrary. □