# Lecture 5: Uniqueness, Convergence in Mean, Completeness

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*"See you at the party, Richter."*
—Douglas Quaid (Hauser)

## 1 Uniqueness

So we have seen conditions when the Fourier series of a function $u$ converges (and to what). Another important question is in what sense we can expect the Fourier coefficients to represent a given function.

**Question**. Suppose that $u, v \in E$ has the Fourier series'

$$u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad \text{and} \quad v(x) \sim \sum_{k=-\infty}^{\infty} d_k e^{ikx}.$$

If $c_k = d_k$ for every $k \in \mathbf{Z}$, what can we say about $u$ and $v$?

We know from before that if $u, v \in E'$ are continuous, then the Fourier series' converge to $u$ and $v$ respectively, so if the Fourier coefficients are the same then the functions are equal. This is *not* true in general, but we will show that equality holds at points where both $u$ and $v$ are continuous (without conditions involving derivatives). To approach this, we need some summation results.

### 1.1 Cesàro Summation

Suppose that $a_1, a_2, a_3, \ldots$ is a sequence of numbers and let $S_n = \sum_{k=1}^{n} a_k$ denote the partial sums. We define

$$\overline{S}_n = \frac{1}{n}\sum_{l=1}^{n} S_l, \quad n = 1, 2, 3, \ldots,$$

to be the mean value of the first $n$ partial sums of the sequence. So yeah, this is a sum of sums. If

$$\lim_{n\to\infty} \overline{S}_n = A$$

exists (in the usual convergent sense), then we say that the sequence $a_1, a_2, a_3, \ldots$ is Cesàro-summable. Note in particular that if $\sum_{k=1}^{\infty} a_k = S$ is convergent, then $A = S$, so we obtain the same answer when doing Cesàro summation. One can see this by considering the following.

Let $S_n \to S$ be convergent and let $\epsilon > 0$. Then there exists $N \in \mathbf{Z}$ such that $|S_m - S| \leq \epsilon$ if $m \geq N$ and

$$\left|\overline{S}_n - S\right| = \left|\frac{1}{n}\sum_{k=1}^{n}(S_k - S)\right| \leq \frac{1}{n}\sum_{k=1}^{m}|S_k - S| + \frac{1}{n}\sum_{k=m+1}^{n}|S_k - S| \leq \frac{1}{n}\sum_{k=1}^{m}|S_k - S| + \epsilon \to \epsilon,$$

as $n \to \infty$. Thus $\overline{S}_n \to S$ as $n \to \infty$.

So why introduce this type of summing? Well, it makes it possible to assign values to series that are classically divergent.

**Example**

Is the sequence $1, -1, 1, -1, 1, -1, \ldots$ Cesàro summable?

**Solution**. The sequence is obviously not summable in the classical sense (why?). However, the answer to the question is yes. Consider the partial sums $S_n$. If $n$ is even, then $S_n = 0$, and if $n$ is odd, then $S_n = 1$. Since

$$\overline{S}_n = \frac{1}{n}\sum_{l=1}^{n} S_l,$$

we obtain that

$$\overline{S}_1 = 1, \quad \overline{S}_2 = \frac{1}{2}, \quad \overline{S}_3 = \frac{2}{3}, \quad \overline{S}_4 = \frac{2}{4} = \frac{1}{2}, \quad \overline{S}_5 = \frac{3}{5}, \quad \overline{S}_6 = \frac{3}{6} = \frac{1}{2}, \quad \overline{S}_7 = \frac{4}{7}, \quad \overline{S}_8 = \frac{4}{8} = \frac{1}{2},$$

and so on. Thus $\overline{S}_{2k} = 1/2$ and $\overline{S}_{2k+1} \to 1/2$ as $n \to \infty$, so $\overline{S} = 1/2$.

Note: this special series is usually referred to as *Grandi's series*.

## 1.2 The Fejér Kernel

Let us look at what happens if we try to perform Cesàro summation for a Fourier series. Working with the complex Fourier series, we define

$$\overline{S}_n(x) = \frac{1}{n+1}\sum_{l=0}^{n} S_l(x) = \frac{S_0(x) + S_1(x) + \cdots + S_n(x)}{n+1},$$

where

$$S_l(x) = \sum_{k=-l}^{l} c_k e^{ikx}, \quad l = 0, 1, 2, \ldots,$$

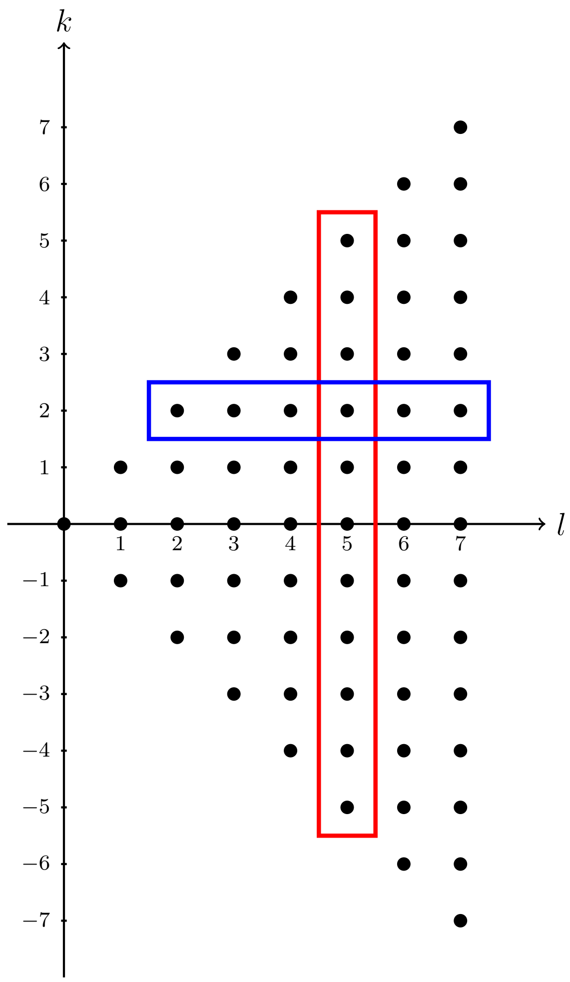
and $c_k$ are the complex Fourier coefficients. The expression for $\overline{S}_n(x)$ is basically the Cesàro mean for the symmetric partial sums. Let us proceed like we did when identifying the Dirichlet kernel:

$$\begin{aligned}\overline{S}_n(x) &= \frac{1}{n+1}\sum_{l=0}^{n}\sum_{k=-l}^{l} c_k e^{ikx} = \frac{1}{n+1}\sum_{l=0}^{n}\sum_{k=-l}^{l}\frac{1}{2\pi}\left(\int_{-\pi}^{\pi} u(t)e^{-ikt}\,dt\right)e^{ikx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\left(\frac{1}{n+1}\sum_{l=0}^{n}\sum_{k=-l}^{l} e^{ik(x-t)}\right)dt.\end{aligned}$$

Isolating the inner parenthesis, we notice that

$$\frac{1}{n+1}\sum_{l=0}^{n}\sum_{k=-l}^{l} e^{ik(x-t)} = \frac{1}{n+1}\sum_{k=-n}^{n} e^{ik(x-t)}\sum_{l=|k|}^{n} 1 = \sum_{k=-n}^{n}\frac{n-|k|+1}{n+1}e^{ik(x-t)}$$
$$= \sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)e^{ik(x-t)},$$

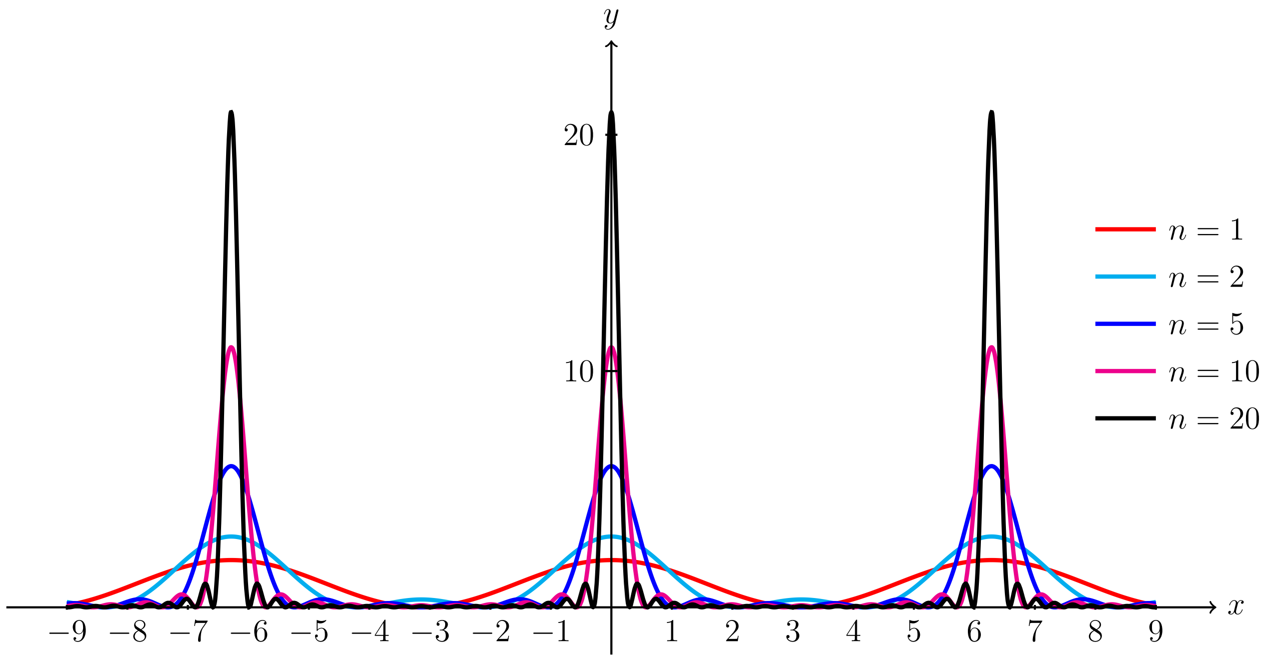
where we changed the order of summation. To see why this looks the way it does, consider the figure below (it's the same type of thinking we did with multiple integrals). Instead of summing over the red rectangles we switch and sum over the blue ones instead.




**The Fejér kernel**

**Definition.** We define the **Fejér kernel** $F_n(x)$ as

$$F_n(x) = \frac{1}{n+1}\sum_{l=0}^{n}\sum_{k=-l}^{l} e^{ikx} = \sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)e^{ikx}, \quad n = 0, 1, 2, \ldots$$




Obviously $F_n(x)$ is an even $2\pi$-periodic function (similar to the Dirichlet kernel) and we can write

$$\overline{S}_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)F_n(x-t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t+x)F_n(t)\,dt.$$

Notice also that $F_n(x)$ is a sum of Dirichlet kernels $D_l(x)$, giving rise to the representation

$$F_n(x) = \frac{1}{n+1}\sum_{l=0}^{n}\frac{\sin((2l+1)x/2)}{\sin(x/2)},\ x \neq 2k\pi,\ k \in \mathbf{Z}.$$

There are more properties of the Fejér kernel that will be important. Let's summarize these.

## Properties of the Fejér kernel

**Theorem.**

(i) $F_n(2k\pi) = n+1$, $k \in \mathbf{Z}$.

(ii) $F_n(x) = \dfrac{1}{n+1}\left(\dfrac{\sin((n+1)x/2)}{\sin(x/2)}\right)^2,\ x \neq 2k\pi,\ k \in \mathbf{Z}.$

(iii) $\displaystyle\int_T F_n(x)\,dx = 2\pi.$

(iv) If $0 < \tau < \pi$, then $F_n \to 0$ uniformly on the set $[-\pi, -\tau] \cup [\tau, \pi]$ as $n \to \infty$.

**Proof**. We obtain the first point by direct verification from the definition of $F_n$. To prove the second identity, observe that

$$
\begin{aligned}
(n+1)\sin(x/2)^2F_n(x) &= -\frac{1}{4}\sum_{l=0}^{n}\left(e^{ix/2}-e^{-ix/2}\right)\left(e^{i(2l+1)x/2}-e^{-i(2l+1)x/2}\right)\\
&= -\frac{1}{4}\sum_{l=0}^{n}\left(e^{i(l+1)x}-e^{-ilx}-e^{ilx}+e^{-i(l+1)x}\right)\\
&= -\frac{1}{4}\sum_{l=0}^{n}\left(2\cos(l+1)x-2\cos lx\right) = \text{/ telescoping sum /}\\
&= -\frac{1}{2}\left(\cos(n+1)x-\cos 0\right) = \frac{1-\cos(n+1)x}{2}\\
&= \sin^2\left((n+1)x/2\right).
\end{aligned}
$$

Furthermore, we see that

$$\int_{-\pi}^{\pi} F_n(x)\,dx = \sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)\int_{-\pi}^{\pi} e^{ikx}\,dx = 2\pi,$$

since $e^{ikx}$ is $2\pi$-periodic when $k \in \mathbf{Z}$ and $k \neq 0$.

The last point is a little more subtle. Looking at the graphs above, we see that the mass seems to be centering more and more around the origin, so we might expect something if we avoid the origin. Indeed, we can see that

$$\|F_n\|_{C[\tau,\pi]} = \frac{1}{n+1}\max_{\tau\le x\le\pi}\left(\frac{\sin((n+1)x/2)}{\sin(x/2)}\right)^2 \le \frac{1}{n+1}\max_{\tau\le x\le\pi}\frac{1}{\sin^2(x/2)} \le \frac{1}{n+1}\frac{1}{\sin^2(\tau/2)} \to 0,$$

so $F_n \to 0$ uniformly on $[\tau,\pi]$. This also implies uniform convergence for $[-\pi,-\tau]$ since $F_n$ is an even function.

**Theorem.** Suppose that $u \in E$. Then

$$\lim_{n\to\infty}\overline{S}_n = \frac{u(x^+)+u(x^-)}{2}$$

for $x \in [-\pi,\pi]$.

**Proof**. This mirrors the proof of the corresponding theorem for $u \in E'$ when we used the Dirichlet kernel. We need to show that

$$\frac{1}{2\pi}\int_0^{\pi}(u(x+t)-u(x^+))F_n(t)\,dt + \frac{1}{2\pi}\int_{-\pi}^{0}(u(x+t)-u(x^-))F_n(t)\,dt \to 0,$$

as $n\to\infty$. This implies that

$$\lim_{n\to\infty}\overline{S}_n = \frac{u(x^+)+u(x^-)}{2}$$

since

$$\frac{1}{2\pi}\int_{-\pi}^{0}F_n(t)\,dt = \frac{1}{2\pi}\int_0^{\pi}F_n(t)\,dt = \frac{1}{2}.$$

Let $\epsilon > 0$. Since $u$ has a right-hand limit at $x$, there is a $\delta > 0$ such that

$$0 < t < \delta \quad \Rightarrow \quad |u(x+t) - u(x^+)| < \epsilon.$$

We exploit this and the uniform convergence of $F_n$ to obtain that

$$\begin{aligned}\left|\frac{1}{2\pi}\int_0^{\pi}(u(x+t)-u(x^+))F_n(t)\,dt\right| &\leq \frac{1}{2\pi}\int_0^{\delta}\epsilon F_n(t)\,dt + \frac{1}{2\pi}\int_{\delta}^{\pi}|u(x+t)-u(x^+)|F_n(t)\,dt \\ &\leq \frac{\epsilon}{2}\frac{1}{\pi}\int_0^{\pi}F_n(t)\,dt + \frac{1}{2\pi}\int_{\delta}^{\pi}|u(x+t)-u(x^+)|F_n(t)\,dt \\ &\to \frac{\epsilon}{2}\end{aligned}$$

as $n \to \infty$ since $F_n$ converges uniformly to zero on $[\delta, \pi]$. The second integral is handled analogously. □

The following corollary is clear since if $S_n(x)$ converges, then $\overline{S}_n(x)$ converges to the same value.

**Corollary.** Suppose that $u \in E$. If the Fourier series is convergent at $x \in [-\pi, \pi]$, then

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = \frac{u(x^+) + u(x^-)}{2}.$$

So basically we could say that "if it converges, it converges correctly" (where it refers to the Fourier series of something in $E$). Furthermore, we have the following uniqueness result for functions in $E$.

**Corollary.** Suppose that $u, v \in E$. If

$$u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad \text{and} \quad v(x) \sim \sum_{k=-\infty}^{\infty} d_k e^{ikx}$$

and $c_k = d_k$ for every $k \in \mathbf{Z}$, then $u(x) = v(x)$ at every point $x \in [-\pi, \pi]$ where both $u$ and $v$ are continuous.

If $u$ is continuous on $[-\pi, \pi]$, then we can use the uniform continuity of $u$ in the proof above to show that $\overline{S_n}(x)$ converges uniformly (for a fixed $\epsilon$ we can use the same $\delta$ for every $x$).

### Uniform convergence

**Corollary.** If $u \in E$ is continuous and $u(-\pi) = u(\pi)$, then $\overline{S_n}(x)$ converges uniformly to $u$.

## 2 $E$, $E'$ and All That Stuff

So we've seen results now that requires different things of the function $u$ to obtain convergence in different senses. To summarize, some of the things we know are the following.

(i) If $u \in E$, then $u$ has a Fourier series (convergence of which is unknown).

(ii) If $u \in E$, then $\overline{S_n}(x) \to \dfrac{u(x^+) + u(x^-)}{2}$.

(iii) If $u, v \in E$ and $\widehat{u}[k] = \widehat{v}[k]$, $k \in \mathbf{Z}$, then $u(x) = v(x)$ whenever $u$ and $v$ are continuous at $x$.

(iv) If $u \in E$ and $D^{\pm}u(x)$ exists, then $S_n(x) \to \dfrac{u(x^+) + u(x^-)}{2}$. If $u \in E'$, this limit holds for all $x$.

(v) If $u' \in E$, $u$ is continuous and $u(-\pi) = u(\pi)$, then $S_n(x)$ converges uniformly to $u(x)$.

(vi) If $u' \in E$ and $u$ is continuous on $[a, b] \subset ]-\pi, \pi[$, then $S_n(x)$ converges uniformly on $[a, b]$.

It is therefore reasonable to question as to whether there are differences between these classes of functions. First, let's take a look at the one-sided derivatives.

**Theorem.** If $u' \in E$ is continuous, then $D^{\pm}u(x) = \lim_{y \to x^{\pm}} u'(y)$.

**Proof**. If $u' \in E$, then $u'$ is piecewise continuous. If $x$ is a point of continuity for $u'$, then $D^{\pm}u(x) = u'(x)$ immediately. If $x$ is a "jump"-point for $u'$, we need to be a bit more careful. Let $h > 0$ and recall the mean value theorem: if $u$ is continuous on $[x, x + h]$ and differentiable on $]x, x + h[$, then there exists a number $\xi$ such that

$$\frac{u(x + h) - u(x)}{h} = u'(\xi), \quad \text{where } x < \xi < x + h.$$

Letting $h \to 0^+$, we find that

$$D^+u(x) = \lim_{h \to 0^+} \frac{u(x + h) - u(x)}{h} = \lim_{h \to 0^+} u'(\xi) = u'(x^+),$$

since $u' \in E$ and $\xi \to x^+$ (we know that the one-sided limit exists since $u' \in E$). The left-hand derivative $D^-u(x)$ is handled analogously. □

The previous theorem does *not* hold if we only know that $u \in E'$. If we don't know that the derivative is continuous on $]x - \delta, x[$ or $]x, x + \delta[$, then we have to use the definition of $D^-u(x)$ and $D^+u(x)$ directly.

**The difference between $u' \in E$ and $u \in E'$**

Recall the main theorem from last lecture. One of the conditions were that $u' \in E$. So what does $u' \in E$ mean? As we saw then, we basically intend for this to mean that the derivative is a piecewise continuous function. What this entails for $u$ is that the two-sided derivative might not exist at some points, but we still write $u' \in E$. The reason for this is that we don't care what actually happens at individual points, but rather the limiting behavior of the function when we approach the point.

If $u \in E'$, then we only know that the function has one-sided derivatives at every point. This is *not* sufficient for $u'$ to be piecewise continuous. In fact $u'$ might be very discontinuous.

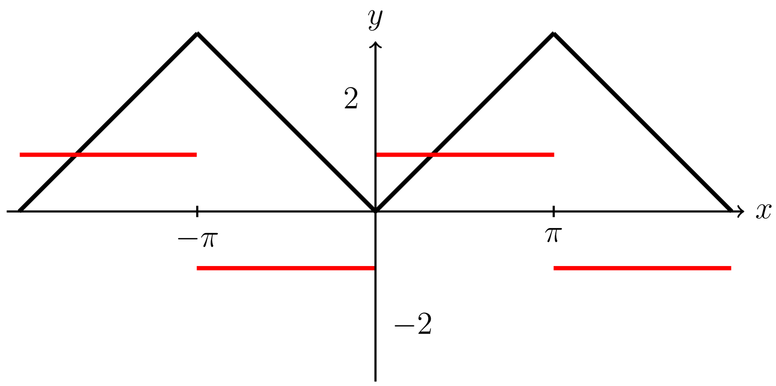
## 2.1 Some Examples

Let's consider some examples that show the differences between the conditions. The black graphs depict the function and the red graphs the derivative.

**Example**

Let $u(x) = |x|$ for $-\pi \le x \le \pi$ and extend periodically. Then $u \in E$ is continuous and $u \in E'$. Moreover, $u' \in E$.

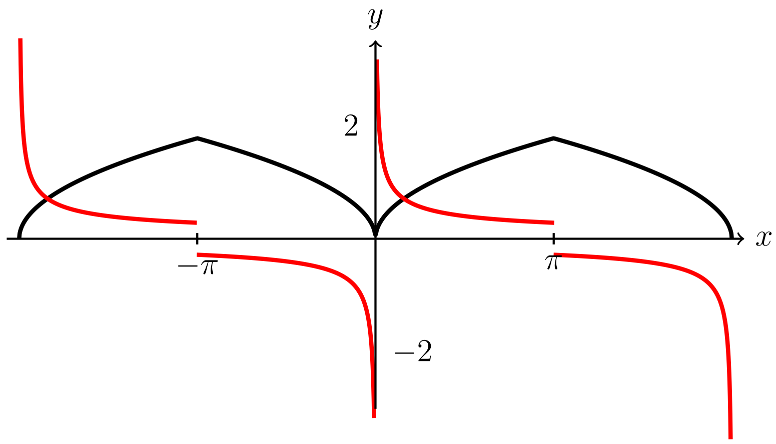
Clearly $u'(x) = -1$ if $-\pi < x < 0$ and $u'(x) = 1$ if $0 < x < \pi$ (and then extend periodically). At $x = 0$, $u'(0)$ does not exist. However, $D^{\pm}u(0) = \pm 1$, and similarly $D^{\pm}(k\pi) = \pm(-1)^k$ for $k \in \mathbf{Z}$ (yeah...).




**Example**

Let $u(x) = \sqrt{|x|}$ for $-\pi \le x \le \pi$ and extend periodically. Then $u \in E$ is continuous but $u \notin E'$.

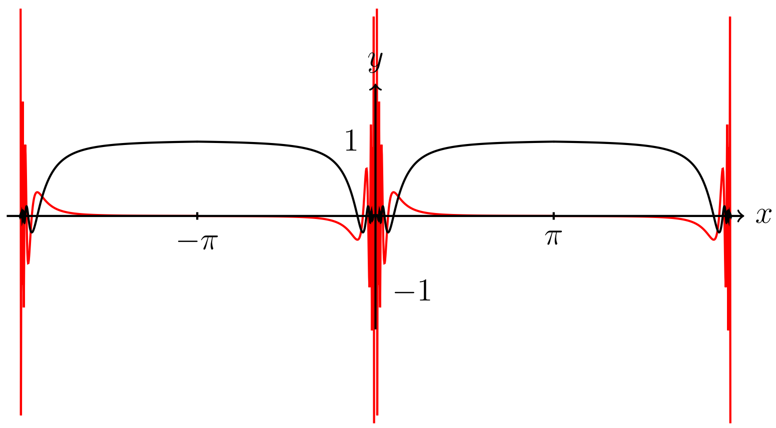
For $-\pi < x < 0$, we find that $u'(x) = -|x|^{-1/2}/2$ and $D^{-}u(0)$ doesn't exist (would be $-\infty$). However, $D^{+}u(-\pi) = -|\pi|^{1/2}/2$. Analogously, for $0 < x < \pi$, we find that $u'(x) = |x|^{-1/2}/2$ and $D^{+}u(0)$ doesn't exist (would be $\infty$). However, $D^{-}u(\pi) = |\pi|^{1/2}/2$.




**Example**

Let $u(x) = x \sin \frac{1}{x}$ for $-\pi \le x \le \pi$ and extend periodically. Then $u \in E$ is continuous but $u \notin E'$. In fact, $D^{\pm}u(0)$ does not even exist if $\pm\infty$ is allowed (this is worse than $\sqrt{|x|}$).

For $-\pi \le x \le \pi$ and $x \ne 0$, it is clear that $u'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$, but $D^{\pm}u(0)$ does not exist (not even if we allow $\pm\infty$ as possibilities). In the graph below, the scale of the function is ten times the size of the derivative.



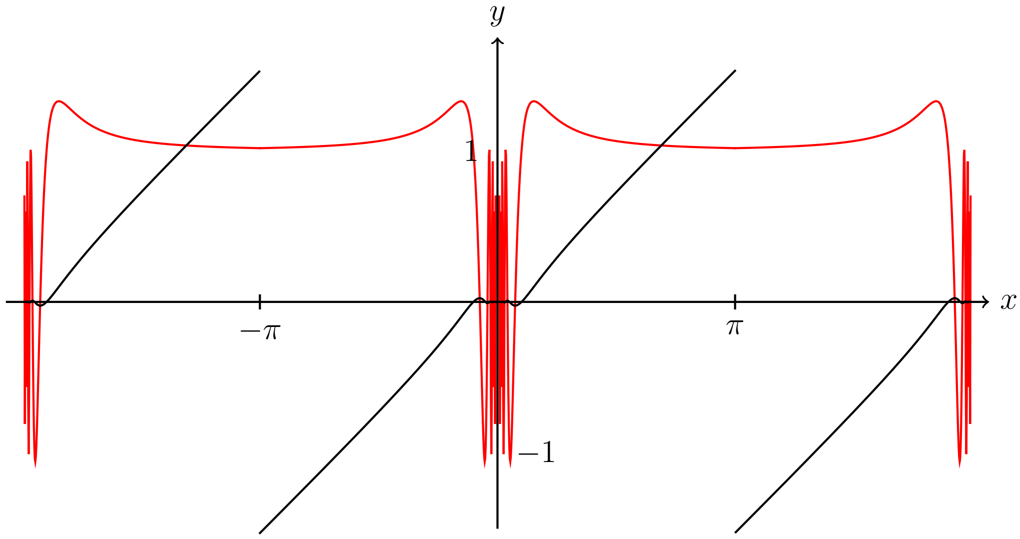

**Example**

Let $u(x) = x^2 \sin \frac{1}{x}$ for $-\pi \le x < 0$ and $0 < x < \pi$. Put $u(0) = 0$ and extend $u$ periodically. Then $u'$ exists everywhere in $]-\pi, \pi[$ and $D^{\pm}u(-\pi)$ and $D^{\pm}u(\pi)$ exists. However, the derivative $u'$ is discontinuous at $x = 0$. Moreover, $u' \notin E$.

For $-\pi < x < \pi$ and $x \ne 0$, it is clear that $u'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$. For $x = 0$, we find that

$$u'(0) = \lim_{h\to 0} \frac{u(h) - u(0)}{h} = \lim_{h\to 0} \frac{1}{h} h^2 \sin \frac{1}{h} = 0 = \lim_{h\to 0} h \sin \frac{1}{h} = 0,$$

so $u'(0)$ exists. Clearly it is not true that $u'(x) \to 0$ as $x \to 0^{\pm}$, so by the theorem we proved earlier (about the limits of $u'$ if $u' \in E$ being the onesided derivatives), it is impossible that $u' \in E$. More directly, consider the limit of $u'(x)$ as $x \to 0^{\pm}$. Neither limit exists, so $u' \notin E$.




**Example**

The Weierstrass function $W(x)$ (look back at the section with the M-test in lecture 3) is a continuous function, so $W \in E$. However, this function is nowhere differentiable going so far that $|D^{\pm}W(x)| = \infty$ at every point. Clearly $W \notin E'$.

## 2.2 How Discontinuous Can a Derivative Be?

So the previous examples (except for the beautiful Weierstrass function) had problems at a single point (and maybe at the endpoints). Obviously we can construct something that has problems at each point of any finite set (which would make the function look quite horrible), but from a mathematical perspective that's usually not that bad (a finite set is rather small compared to an interval). Could we have problems at an infinite set of points? At all points?

Let's recall a famous theorem by Darboux, claiming that the derivative of a differentiable function has the intermediate value property.

**Darboux's theorem**

**Theorem.** Suppose that $u$ is differentiable on $[a, b]$ and that $u'(a) < u'(b)$. If $\lambda$ is a number such that $u'(a) < \lambda < u'(b)$, then there exists a point $c \in ]a, b[$ such that $u'(c) = \lambda$.

**Proof.** We want to prove that there exists some $c \in ]a, b[$ such that $u'(c) - \lambda = 0$. Let's define $U(x) = u(x) - \lambda x$ so that $U'(c) = u'(c) - \lambda$. Then $U'(a) = u'(a) - \lambda < 0$. Hence there's some point $x_0 > a$ such that $U(x_0) < U(a)$. Similarly, since $U'(b) = u'(b) - \lambda > 0$, there's some point $x_1 < b$ such that $U(x_1) < U(b)$.

What this means, is that the minimum of $U$ on $[a, b]$ is *not* attained at the endpoints. With $U$ being a continuous function and $[a, b]$ being compact, we do however know that the minimum is attained. This ensures the existence of a point $c \in ]a, b[$ such that $U(c)$ is an extreme value and since $U$ is differentiable, this proves that $U'(c) = 0$. □

So what's the use of this result? For one thing, we can show that certain functions *can't* be the derivative of something else. Indeed, as an example consider the function $u(x) = 1$ if $x$ is irrational and $u(x) = 0$ if $x$ is rational. This is a severely discontinuous function. Assuming that $u$ is the derivative of some function $U$, it would follow from Darboux's theorem that $u$ has the intermediate value property. This is obviously false since we can choose any number $\lambda \in ]0, 1[$ where we can't find any $c$ such that $u(c) = \lambda$.

So in other words, if a function is differentiable, then the derivative can't be as bad as this. However, there are differentiable functions where the set of of discontinuities of the derivative is uncountable so it's still pretty bad. In fact, there are functions whose derivatives are so bad that you can't integrate the derivative using the Riemann integral.

## 3 The ON-system $\{e^{ikx}\}_{k\in\mathbf{Z}}$ is Closed in $E$

We will now prove that the ON-system $\{e^{ikx}\}_{k\in\mathbf{Z}}$ is closed in $E$, meaning that we need to show that for every $u \in E$, there is a sequence of constants $c_k \in \mathbf{C}$, $k = 0, 1, 2, \ldots$ such that

$$\lim_{n\to\infty} \left\| u(x) - \sum_{k=-n}^{n} c_k e^{ikx} \right\|_2 = \lim_{n\to\infty} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| u(x) - \sum_{k=-n}^{n} c_k e^{ikx} \right|^2 dx \right)^{1/2} = 0. \quad (1)$$

Note that this result will imply that the Fourier series of $u \in E$ will converge to $u$ in the sense of the norm we use on $E$ (the $L^2$-norm). This is sometimes called *convergence in mean.*

To obtain this result, we need a sequence of approximation results rather typical for (hard) analysis. Recalling from the previous section that we can approximate any continuous function (with derivative in $E$) on $[-\pi, \pi]$ uniformly by the trigonometric polynomial that is its Fourier series (assuming the function has the same value at the endpoints), we need to first approximate $u \in E$ with something continuous.

The procedure will be as follows. We fix some $u \in E$. Next we choose a piecewise constant function $h$ such that

$$\|u - h\|_2 < \frac{\epsilon}{3}.$$

Next we approximate this piecewise constant function $h$ by a piecewise linear[1] continuous function $f$ (satisfying $f(-\pi) = f(\pi)$) such that

$$\|h - f\|_2 < \frac{\epsilon}{3}.$$

Now, since $f$ is continuous and $f' \in E$, we know that the Fourier series of $f$ converges to $f$ uniformly on $[-\pi, \pi]$. This means that we can choose $N$ so that

$$\left\| f(x) - \sum_{k=-n}^{n} c_k e^{ikx} \right\|_2 < \frac{\epsilon}{3}, \quad \text{for } n > N,$$

if $c_k$ are the Fourier coefficients of $f$. Finally, by the triangle inequality we have now obtained that

$$\left\| u(x) - \sum_{k=-n}^{n} c_k e^{ikx} \right\|_2 \leq \|u - h\|_2 + \|h - f\|_2 + \left\| f(x) - \sum_{k=-n}^{n} c_k e^{ikx} \right\|_2 < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

when $n \geq N$, which is precisely what (1) means.

[1]Piecewise linear means that the function is of the form $y = kx + m$ on each "piece."

### 3.1 Approximations...

So the idea and steps were explained in the previous section, but let's take a closer look at the first two approximations (the last one is quite clear) to make sure everything is possible..

First, since $u \in E$ it is Riemann integrable (see TATA41) the following must hold. For every $\epsilon > 0$, there is a *partition* of $[-\pi, \pi]$,

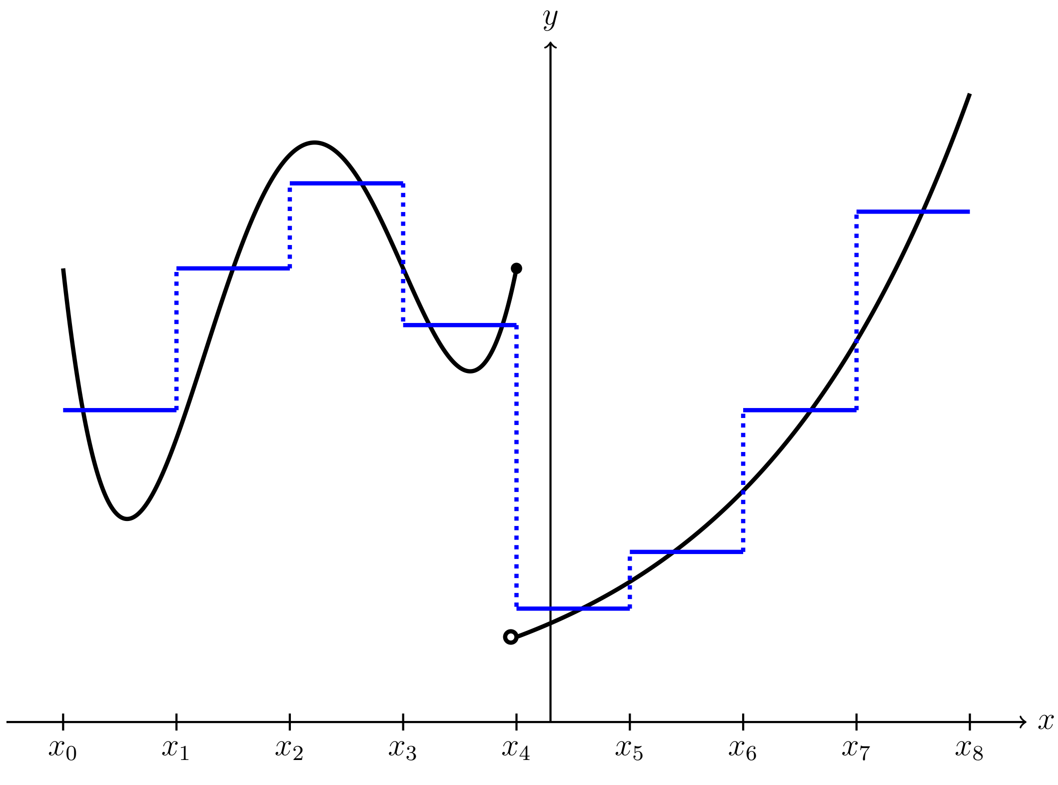
$$x_0 = -\pi < x_1 < x_2 < \cdots < x_n = \pi,$$

and numbers $\xi_i \in [x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$, such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(x) - h(x)|^2 \, dx < \frac{\epsilon^2}{4},$$

where we define the function $h(x)$ to be equal to $d_k = u(\xi_k)$ if $x_k < x \leq x_{k+1}$. Note that $h$ is a piecewise constant function that approximates $u$. See the blue graph below.

We make sure to include the points where $u$ is discontinuous (of which there are a finite number) in the set $\{x_0, x_1, \ldots, x_n\}$, so that $u$ is continuous on each interval $[x_i, x_{i+1}]$ after possible redefinition at the endpoints (remember that the right- and lefthand limits of $u$ exists if $u \in E$).




To see why this is possible, note that the restriction of $u$ to intervals $[a_i, a_{i+1}]$ (after possible redefinition at a finite number of points $a_i$) is uniformly continuous on each $[a_i, a_{i+1}]$. Thus, for any $\epsilon > 0$, there is a $\delta_i > 0$ such that

$$x, y \in [a_i, a_{i+1}] : |x - y| < \delta_i \quad \Rightarrow \quad |u(x) - u(y)| < \frac{\epsilon}{3}.$$

Let $\delta = \min\{\delta_i\}$. Clearly $\delta > 0$, so it is possible to choose a partition $\{x_i\}_{i=0}^{n}$ of $[-\pi, \pi]$ such that $|x_{i+1} - x_i| < \delta$, $i = 0, 1, 2, \ldots, n-1$, and each point $a_i$ can be found in the set $\{x_i\}_{i=0}^{n}$. By the uniform continuity on each $[x_i, x_{i+1}]$, it is clear that

$$|u(x) - h(x)|^2 = |u(x) - d_k|^2 \leq \frac{\epsilon^2}{9}, \quad x_i < x < x_{i+1},$$

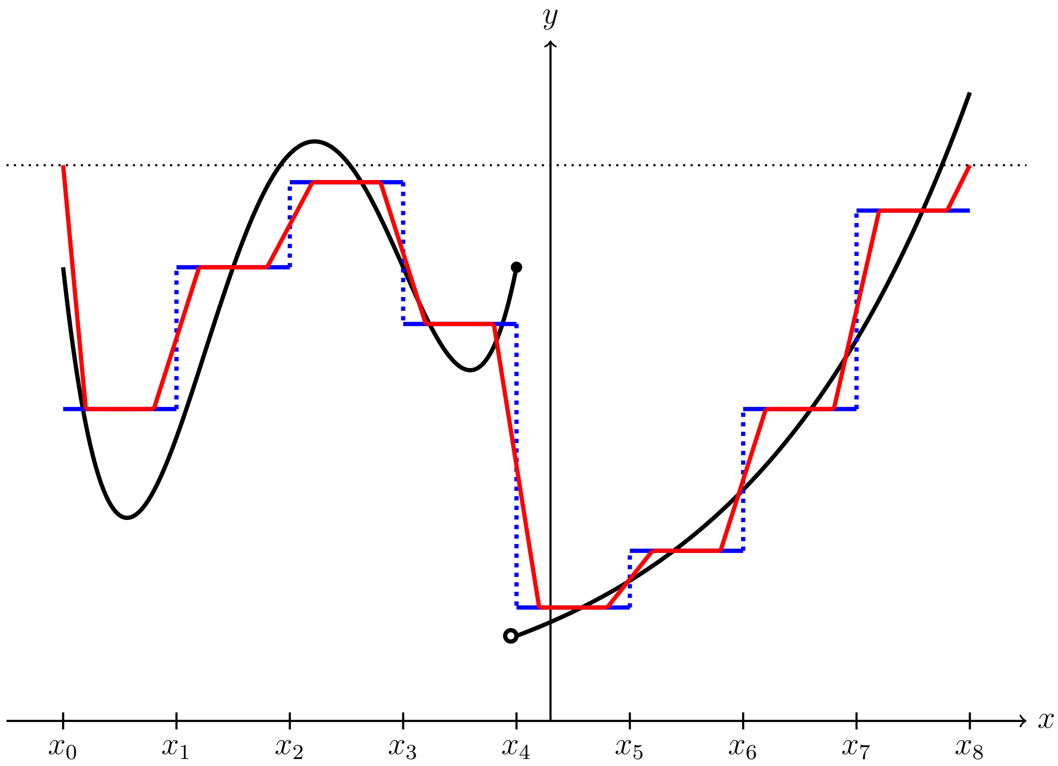
since $d_k = u(\xi_k)$ for some $\xi_k$ such that $x_k < \xi_k \leq x_{k+1}$. The inequality might not hold at the end-points, but this does not matter for the integral. This implies that

$$\int_{x_i}^{x_{i+1}} |u(x) - h(x)|^2 \, dx \leq \frac{\epsilon^2}{9}|x_{i+1} - x_i|, \quad i = 0, 1, 2, \ldots, n-1,$$

so

$$\|u - h\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(x) - h(x)|^2 \, dx = \frac{1}{2\pi} \left( \sum_{k=0}^{n-1} \int_{x_i}^{x_{i+1}} |u(x) - h(x)|^2 \, dx \right)$$
$$\leq \frac{1}{2\pi} \left( \sum_{k=0}^{n-1} \frac{\epsilon^2}{9} |x_{i+1} - x_i| \right) = \frac{\epsilon^2}{9}.$$

Next step is to approximate $h$ by a continuous function $f$ such that $f(-\pi) = f(\pi)$. To this end, choose a $\delta > 0$ such that $\delta < \frac{\pi\epsilon^2}{36M^2 n}$ (yeah.. we'll get to that), where $M$ is some number such that $|h(x)| \leq M$ for all $x$. Define $f$ such that $f(x) = d_k$ when $x_k + \delta \leq x \leq x_{k+1} - \delta$ and between these intervals, a straight line that connects the $y$-values $d_k$ with $d_{k+1}$. At the endpoints, we connect $d_0$ and $d_n$ with the $y$-value that is the mean value of $u(-\pi)$ and $u(\pi)$. See the red graph below.




The function $f$ is continuous on $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$. We extend $f$ periodically to $\mathbf{R}$.

Since $f = h$ on large chunks of $[-\pi, \pi]$, we now note that

$$\begin{aligned}
\|f-h\|_2^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)-h(x)|^2\,dx \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{-\pi+\delta} |f(x)-h(x)|^2\,dx + \sum_{k=1}^{n-1}\int_{x_k-\delta}^{x_k+\delta} |f(x)-h(x)|^2\,dx + \int_{\pi-\delta}^{\pi} |f(x)-h(x)|^2\,dx\right) \\
&\le \frac{1}{2\pi}\left(4M^2(\delta + (n-1)\cdot 2\delta + \delta)\right) = \frac{4M^2 n\delta}{\pi} \\
&\le \frac{\epsilon^2}{9},
\end{aligned}$$

where we used the rough estimate $|f(x)-h(x)| \le 2M$ which holds if $|f(x)| \le M$ (which implies that $|h(x)| \le M$ as well). Note that $f' \in E$.

# 4 Parseval's Formula

Recall from Lecture 2 that Parsevals's identity holds for closed ON systems (and we just proved this in the case of $E$):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|^2\,dx = \sum_{k=-\infty}^{\infty} |c_k|^2,$$

where

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx, \quad k \in \mathbf{Z}.$$

Furthermore, this could be generalized as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\overline{v(x)}\,dx = \sum_{k=-\infty}^{\infty} c_k\overline{d_k},$$

where

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx \quad \text{and} \quad d_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(x)e^{-ikx}\,dx, \quad k \in \mathbf{Z}.$$

**Example**

Calculate $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

**Solution**. Note that $u(x) = x$, $-\pi \le x < \pi$, has the Fourier coefficients $c_k = i(-1)^k/k$ for $k \neq 0$ and $c_0 = 0$ (show this). Hence

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = 2\sum_{k=1}^{\infty} \frac{1}{k^2}$$

and by Parseval's identity this series is equal to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|^2\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^3}{6\pi} = \frac{\pi^2}{3},$$

so

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

This is one way of proving this famous formula.

## 4.1 Parseval's Formula in the Real Case

The corresponding formula for a real Fourier series

$$u(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \big(a_k \cos kx + b_k \sin kx\big)$$

is given by

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |u(x)|^2\, dx = \frac{|a_0|^2}{2} + \sum_{k=1}^{\infty} \big(|a_k|^2 + |b_k|^2\big).$$

Notice in particular the normalizing constants (this is a common source for mistakes). If we have a Fourier series already expanded in the real form, it is usually easier to use the identity above than to rewrite the Fourier series in complex form. It might also be easier if we exploit eveness or oddness of the integrands. Another possible feature is that we usually obtain a series that's only infinite in one direction.

**Example**

Find $\sum_{k=1}^{\infty} \frac{1}{k^6}$. *Hint: consider* $u(x) = x^3 - \pi^2 x$.

**Solution**. Considering the hint (and the chapter we're in...), let's find the Fourier coefficients of $u$. It is an odd function and if we restrict $u$ to $[-\pi, \pi]$ and extend periodically, we obtain a $2\pi$-periodic function such that $u'(-\pi) = u'(\pi)$. We could exploit this fact to expand $u'$ instead of $u$, obtaining a uniformly convergent Fourier series for $u'$ and then integrating this series (which is allowed due to the uniform convergence) to obtain the Fourier series for $u$. Unfortunately we can't repeat this argument for $u''$ ($u''(-\pi) \neq u''(\pi)$), so it's not really worth the headache. Let's just bite the bullet and do some integration by parts. Since $u$ is odd, $a_k = 0$ for $k \geq 0$. For $k \geq 1$, we find that

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} (x^3 - \pi^2 x)\sin kx\, dx = \underbrace{\left[-\frac{(x^3-\pi^2 x)\cos kx}{\pi k}\right]_{-\pi}^{\pi}}_{=0 \text{ since } u(\pm\pi)=0} + \int_{-\pi}^{\pi} \frac{(3x^2-\pi^2)\cos kx}{\pi k}\, dx \\
&= \underbrace{\left[\frac{(3x^2-\pi^2)\sin kx}{\pi k^2}\right]_{-\pi}^{\pi}}_{=0 \text{ since } u'(\pm\pi)=0} - \int_{-\pi}^{\pi} \frac{6x \sin kx}{\pi k^2}\, dx = \left[\frac{6x\cos kx}{\pi k^3}\right]_{-\pi}^{\pi} - \underbrace{\int_{-\pi}^{\pi} \frac{6\cos kx}{\pi k^3}\, dx}_{\substack{=0 \text{ since we have} \\ k \text{ periods of } \cos kx.}} \\
&= \frac{12(-1)^k}{k^3}.
\end{aligned}$$

Since $a_k = 0$, $k \geq 0$, we obtain from Parseval's formula that

$$\frac{|a_0|^2}{2} + \sum_{k=1}^{\infty} \left(|a_k|^2 + |b_k|^2\right) = \sum_{k=1}^{\infty} |b_k|^2$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} |x^3 - \pi^2 x|^2 \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(x^6 - 2\pi^2 x^4 + \pi^4 x^2\right) dx = \frac{16\pi^6}{105}.$$

So using this equality, we see that

$$\sum_{k=1}^{\infty} |b_k|^2 = 144 \sum_{k=1}^{\infty} \frac{1}{k^6} \quad \Rightarrow \quad \sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{\pi^6}{945}.$$

Could you figure out a way to find $\sum_{k=1}^{\infty} \frac{1}{n^8}$?