Lecture 6: The Fourier Transform

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"Crom! Grant me revenge. And if you're not listening, to hell with you!"
—Conan

1 The Fourier Transform

Formally, we can consider the **Fourier transform** of a function $u\colon \mathbf{R} \to \mathbf{C}$ given by

$$\mathcal{F}\, u(\omega) = \int_{-\infty}^{\infty} u(x) e^{-i\omega x}\, dx, \quad \omega \in \mathbf{R},$$

when this integral exists. When is this the case? Well, if $u \in L^1(\mathbf{R})$ then this integral will be absolutely convergent since $|u(x)e^{-i\omega x}| \leq |u(x)|$ (for ω real) so

$$|\mathcal{F}\, u(\omega)| \leq \int_{-\infty}^{\infty} |u(x)|\, dx < \infty.$$

Note that this bound is uniform in ω, so we have actually proved that

$$\|\mathcal{F}\, u\|_\infty \leq \|u\|_{L^1(\mathbf{R})},$$

meaning that the Fourier transform maps functions from $L^1(\mathbf{R})$ into $L^\infty(\mathbf{R})$. The space $L^1(\mathbf{R})$ will be to hard for us to handle properly though, so let's consider piecewise continuous functions similarly with how we handled Fourier series. In some cases you'll see that $\omega = 2\pi f$ is used. This is to obtain results in terms of frequency (not angular frequency) with the unit Hertz. This won't happen very often in this course, but is quite common in signal processing.

The space $G(\mathbf{R})$

Definition. We define the space $G(\mathbf{R})$ (or just G if the domain is clear from the context) to consist of all piecewise continuous functions $u\colon \mathbf{R} \to \mathbf{C}$ that are absolutely integrable. A function is called piecewise continuous on $\mathbf{R}$ if there is a finite number of exception points in each finite interval $[a, b]$ (meaning that $u \in E[a, b]$).

Note that this means that a function in G might have an infinite number of discontinuity points (but still countably many). A simple example is the *integer function* $u(x) = [x]$ that maps a real value x to its integer part (clearly this function is not integrable however).

When dealing with Fourier transforms, there's some slight variations in the notation. The most common ways to denote the Fourier transform of $u\colon \mathbf{R} \to \mathbf{C}$ are

$$U(\omega) = \widehat{u}(\omega) = \mathcal{F}\, u(\omega).$$

Choose which one you prefer and try to stay consistent (I probably won't..). Note that there are certain instances where a certain notation makes things easier to read, so some variation is alright.

When using $\mathcal{F}\,u(\omega)$, observe that this means that *the function* $\mathcal{F}\,u$ has the argument ω. If we wish to be very careful, we sometimes write $\mathcal{F}(u(x))(\omega)$ to indicate that u is a function of x and the Fourier transform of u is a function of ω, even if this notation is slightly incorrect (u is the function and $u(x)$ is the value of the function at x). We might even write $(\mathcal{F}(u(x)))(\omega)$ if it helps to make something clearer, but severely clumsy notation is a bit like pissing your pants when it's cold (most often it's not the best idea...).

Normalizing constants

There are several, different competing "versions" of the Fourier transform that differs by a constant. In the book the Fourier transform is given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)e^{-i\omega x}\,dx$$

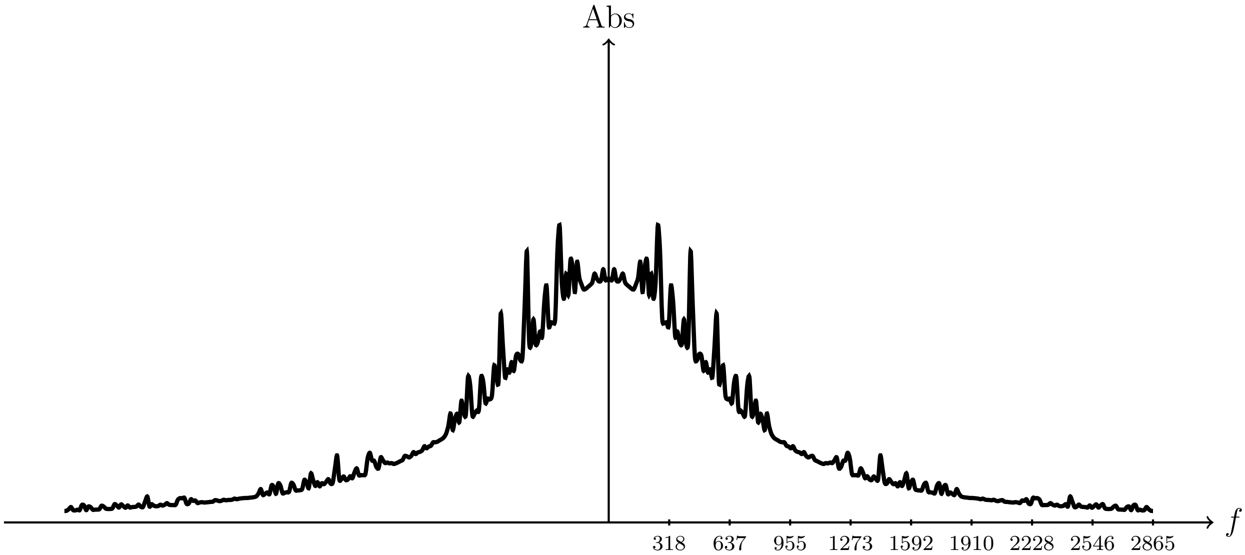
and in other material you might find that the Fourier transform is defined by

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x)e^{-i\omega x}\,dx.$$

The theory will look the same, but obviously the Fourier transforms of specific functions will have different constants attached. Be *very* careful when reading tables! This problem will also return next lecture when we discuss the inverse Fourier transform.

2 Time/Space and Frequency; The Spectrum

We often think of the function $u\colon \mathbf{R} \to \mathbf{C}$ as a function of time or space, meaning that we have values at certain times or at certain points. Taking the Fourier transform of u produces a function $U\colon \mathbf{R} \to \mathbf{C}$, and we consider $U(\omega)$ as a function of angular frequency ω. If we plot the magnitude of U (that is we plot the absolute value), we typically obtain something like this.



Why this example? Why these numbers? Why does the graph look symmetric around the y-axis? So many questions. The connection between the angular frequency and regular frequency is given by

$$\omega = 2\pi f,$$

where f is the regular frequency (measured in Hertz). For an audio signal, we typically consider frequencies below 22 kHz so that's the reason for those numbers. Furthermore, a real-valued function always has a symmetric spectrum. So that's the reason for the symmetry. We'll prove that later on (it's not that difficult). For this reason we usually only plot half of the magnitude spectrum in the case when the signal is real.

The graph depicts the magnitude of the frequency content of the function u. At each ω, we find how much of that frequency that's included in u. You're going to see a lot of this when studying courses in signal processing.

3 Examples

A lot of calculations to derive the Fourier transforms of given functions are rather difficult in that they involve techniques that aren't available to us (like residue calculus from complex analysis). Other problems arrive from our choice of *domain* for the Fourier transform, that is, the space $G(\mathbf{R})$. Not only are we requiring functions to be piecewise continuous, but also absolutely integrable. For example, could we assign a Fourier transform to a non-zero constant? We could, but that basically requires *distribution theory* (and the answer is basically the *Dirac "function"*). So what this means is that we're going to see tables where Fourier transforms are listed that might not be completely in line with what we're able to prove, but we will use these anyway if needed (well.. maybe not, we'll see). Be aware though that we have not covered the necessary theory in that case.

So let's consider some examples where we can actually derive the Fourier transform without any issues.

Example

Show that the Fourier transform of $u(x) = e^{-|x|}$, $x \in \mathbf{R}$, is given by $U(\omega) = \dfrac{2}{1+\omega^2}$.

Solution. We note that $u \in G(\mathbf{R})$ and

$$\int_{-\infty}^{\infty} e^{-|x|}e^{-i\omega x}\,dx = \int_{-\infty}^{0} e^{x(1-i\omega)}\,dx + \int_{0}^{\infty} e^{-x(1+i\omega)}\,dx = \left[\frac{e^{x(1-i\omega)}}{1-i\omega}\right]_{-\infty}^{0} + \left[-\frac{e^{-x(1+i\omega)}}{1+i\omega}\right]_{0}^{\infty}$$
$$= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{1+i\omega+(1-i\omega)}{(1+i\omega)(1-i\omega)} = \frac{2}{1+\omega^2}.$$

We also note the following partial result from the previous calculation.

Example

The Fourier transform of $u(x) = e^{-x}$, $x \geq 0$, is given by $U(\omega) = \dfrac{1}{1+i\omega}$.

We see that in situations where functions are defined from a certain point onward, the following function can be helpful in writing down such expressions.

The Heaviside Function

Definition. The **Heaviside function** H is defined by $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x \geq 0$.

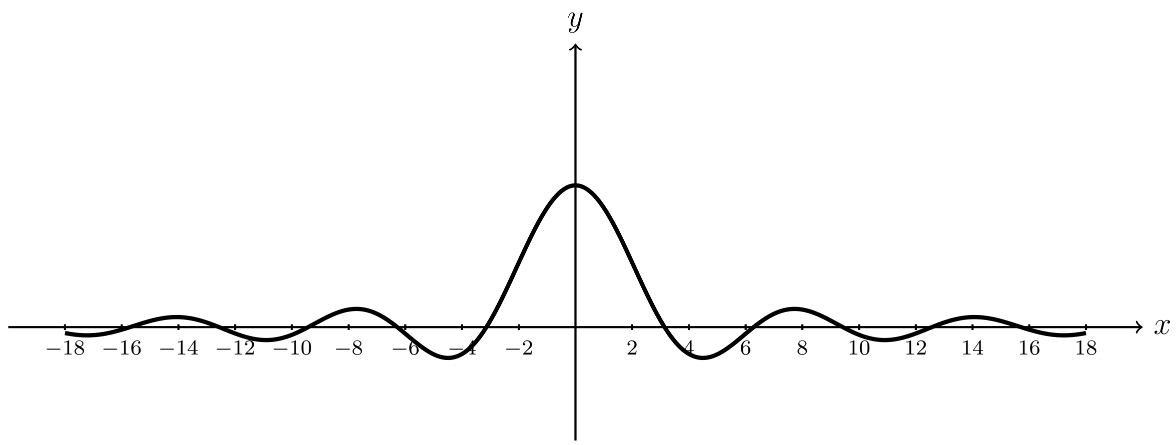
The sinc function

Definition. We define the **sinc**-function by

$$\operatorname{sinc}(x) = \frac{\sin x}{x}, \quad x \neq 0,$$

and $\operatorname{sinc}(0) = 1$ (why?).

The sinc-function is a sinusoid that decays as $1/x$. As we shall see, it is also an important function when dealing with Fourier transforms.



Example

Show that the Fourier transform of $u(x) = 1$, $x \in [-1, 1]$, and $u(x) = 0$ elsewhere, is given by $U(\omega) = 2 \operatorname{sinc} \omega$.

Solution. We note that $u \in G(\mathbf{R})$ and that

$$\int_{-\infty}^{\infty} u(x)e^{-i\omega x}\, dx = \int_{-1}^{1} e^{-i\omega x}\, dx = \left[\frac{e^{-i\omega x}}{-i\omega}\right]_{-1}^{1} = \frac{e^{i\omega} - e^{-i\omega}}{i\omega} = 2 \operatorname{sinc} \omega, \quad \omega \neq 0.$$

For $\omega = 0$, we find that $\mathcal{F} u(0) = 2$. This is $2 \operatorname{sinc}(0)$ so the Fourier transform of the "box" is continuous also at the origin. We will show that the continuity of the Fourier transform is true for any $u \in G(\mathbf{R})$.

Note also the contrast between the graphs of the function and its Fourier transform. Indeed, the Fourier transform (while decaying) is oscillating around the ω-axis all the way to infinity, whereas the function u is extremely limited with respect to x (it's equal to zero outside $[-1, 1]$). This is an intrinsic property of the Fourier transform. We can't have something that's both limited in x and ω at the same time. You're going to see this phenomenon in a lot of applied settings ranging from quantum mechanics (hello Heisenberg) to telecommunication.

4 Properties of the Fourier Transform

In the previous examples, we saw that a real valued function might give both real and complex valued Fourier transforms, but in the case when the function was symmetric we obtained a real valued transform. Is this true in general? Or was there something else that happened in these examples that produced the result? Or was it just coincidence? These types of symmetry questions and general properties of the Fourier transform are important and also what enables us to develop useful concise tables that work together with certain rules. So let's take a look at the properties and rules of the Fourier transform.

Theorem. For $u \in G$, the Fourier transform $\mathcal{F}u$ is uniformly continuous on $\mathbf{R}$.

Proof. So... there's an easy way of doing this by means of the Lebesgue dominated convergence theorem. However, this is slightly outside the course, so let's try something else. Let $U(\omega)$ be the Fourier transform of $u \in G$ and let h be a small real number. Then

$$|U(\omega+h) - U(\omega)| = \left| \int_{-\infty}^{\infty} u(x) \left(e^{-i(\omega+h)x} - e^{-i\omega x} \right) dx \right| \leq \int_{-\infty}^{\infty} |u(x)| \left| e^{-i(\omega+h)x} - e^{-i\omega x} \right| dx.$$

Now, we need to do something with the difference of complex exponentials. A rough estimate is given by

$$\left| e^{-i(\omega+h)x} - e^{-i\omega x} \right| \leq \left| e^{-i(\omega+h)x} \right| + \left| e^{-i\omega x} \right| = 2$$

so at least it is bounded. However, clearly the difference also goes to zero as $h \to 0$, so we can do better. Indeed, let $\alpha, \beta \in \mathbf{R}$. Then

$$\begin{aligned} \left| e^{i\alpha} - e^{i\beta} \right|^2 &= |\cos\alpha + i\sin\alpha - \cos\beta - i\sin\beta|^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 \\ &= \cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &= 2(1 - \cos(\alpha-\beta)) = 4\sin^2\left(\frac{\alpha-\beta}{2}\right) \leq 4\left(\frac{\alpha-\beta}{2}\right)^2 = (\alpha-\beta)^2, \end{aligned}$$

since $|\sin x| \leq |x|$ for $x \in \mathbf{R}$. This implies that

$$\left| e^{-i(\omega+h)x} - e^{-i\omega x} \right| \leq |-(\omega+h)x + \omega x| = |h||x|.$$

Let $\epsilon > 0$. We will prove that there exists $\delta > 0$ such that

$$|h| < \delta \quad \Rightarrow \quad |U(\omega+h) - U(\omega)| < \epsilon \text{ for every } \omega \in \mathbf{R}. \tag{1}$$

Since u is absolutely integrable, there exists some number $R > 0$ such that

$$\int_{|x|>R} |u(x)|\, dx < \frac{\epsilon}{4}.$$

Then

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x)| \left| e^{-i(\omega+h)x} - e^{-i\omega x} \right| dx &= \int_{|x|>R} |u(x)| \left| e^{-i(\omega+h)x} - e^{-i\omega x} \right| dx \\ &\quad + \int_{-R}^{R} |u(x)| \left| e^{-i(\omega+h)x} - e^{-i\omega x} \right| dx \\ &\leq 2\int_{|x|>R} |u(x)|\, dx + \int_{-R}^{R} |u(x)||h||x|\, dx \\ &\leq \frac{2\epsilon}{4} + |h|R\int_{-R}^{R} |u(x)|\, dx \leq \frac{\epsilon}{2} + |h|R\int_{-\infty}^{\infty} |u(x)|\, dx, \end{aligned}$$

so we can choose $\delta = \dfrac{\epsilon}{2R\|u\|_{L^1}}$ to obtain (1). □

The Riemann-Lebesgue "Lemma"

Theorem. For $u \in G$ we have $\mathcal{F}\,u(\omega) \to 0$ as $|w| \to \infty$.

Proof. Let $\epsilon > 0$. We prove that there exists $N > 0$ such that

$$|\omega| > N \quad \Rightarrow \quad |U(\omega)| < \epsilon. \tag{2}$$

Since u is absolutely integrable, there exists $M > 0$ such that

$$\int_{|x|>M} |u(x)|\, dx < \frac{\epsilon}{3}. \tag{3}$$

Since u is Riemann integrable on $[-M, M]$, there exists a step function h such that

$$\int_{-M}^{M} |u(x) - h(x)|\, dx < \frac{\epsilon}{3} \tag{4}$$

(we could take the lower sum for instance so that $|u(x) - h(x)| = u(x) - h(x)$). Let the step function be equal to the constant c_k when $x_k < x < x_{k+1}$, $k = 0, 1, \ldots, m-1$, where

$$-M = x_0 < x_1 < x_2 < \cdots < x_m = M$$

is a suitable partition of $[-M, M]$. Observe now that for $\omega \neq 0$,

$$\begin{aligned}\left|\int_{x_k}^{x_{k+1}} h(x)e^{-i\omega x}\, dx\right| &= \left|\int_{x_k}^{x_{k+1}} c_k e^{-i\omega x}\, dx\right| = \left|c_k \left[\frac{e^{-i\omega x}}{-i\omega}\right]_{x=x_k}^{x=x_{k+1}}\right| \le \frac{|c_k|}{|\omega|}\left|e^{-i\omega x_{k+1}} - e^{-i\omega x_k}\right| \\ &\le \frac{2|c_k|}{|\omega|}.\end{aligned}$$

Hence

$$\left|\int_{-M}^{M} h(x)e^{-i\omega x}\, dx\right| \le \sum_{k=0}^{m-1}\left|\int_{x_k}^{x_{k+1}} h(x)e^{-i\omega x}\, dx\right| \le \frac{2}{|\omega|}\sum_{k=0}^{m-1}|c_k|.$$

Let $N > 6\epsilon^{-1}\displaystyle\sum_{k=0}^{m-1}|c_k|$. Then, if $|w| > N$, we have

$$\left|\int_{-M}^{M} h(x)e^{-i\omega x}\, dx\right| < \frac{\epsilon}{3}. \tag{5}$$

By equations (3), (4) and (5), we obtain

$$\begin{aligned}|U(\omega)| &= \left|\int_{-\infty}^{\infty} u(x)e^{-i\omega x}\, dx\right| \\ &\le \int_{|x|>M} |u(x)|\, dx + \left|\int_{-M}^{M} (u(x) - h(x))e^{-i\omega x}\, dx\right| + \left|\int_{-M}^{M} h(x)e^{-i\omega x}\, dx\right| \\ &< \frac{\epsilon}{3} + \int_{-M}^{M} |u(x) - h(x)|\, dx + \frac{\epsilon}{3} < \epsilon,\end{aligned}$$

which is (2). □

This result also implies the following useful result.

Corollary. If $u \in E[a, b]$ (piecewise continuous on $[a, b]$ and integrable), then

$$\lim_{M\to\pm\infty} \int_a^b u(x)\sin(Mx) = 0 \quad \text{and} \quad \lim_{M\to\pm\infty} \int_a^b u(x)\cos(Mx) = 0.$$

5 Rules for the Fourier Transform

Suppose throughout that $u, v \in G(\mathbf{R})$. Additional assumptions will be stated in the theorems.

Linearity

Theorem. If a, b are constants, then $\mathcal{F}(au + bv) = a\,\mathcal{F}\,u + b\,\mathcal{F}\,v$.

Proof. This follows from the linearity of the integral defining the Fourier transform.

Scaling

Theorem. If $a \neq 0$, then $\mathcal{F}(u(ax))(\omega) = \dfrac{1}{|a|}\,\mathcal{F}(u(x))\left(\dfrac{\omega}{a}\right)$.

Proof. First, assume that $a > 0$. Observing that

$$\mathcal{F}(u(ax))(\omega) = \int_{-\infty}^{\infty} u(ax)e^{-i\omega x}\,dx = \Big/ \; y = ax \; \Big/ = \int_{-\infty}^{\infty} u(y)e^{-i\omega y/a}\,\frac{dy}{a}$$
$$= \frac{1}{a}\int_{-\infty}^{\infty} u(y)e^{-i(\omega/a)y}\,dy = \frac{1}{a}\,\mathcal{F}\,u\left(\frac{\omega}{a}\right).$$

If $a < 0$, then we need to note that when doing the substitution, the limits will exchange places (so the integral goes from $+\infty$ to $-\infty$). Changing this back changes the sign of the integral, so we obtain that

$$\mathcal{F}(u(ax))(\omega) = -\frac{1}{a}\,\mathcal{F}\,u\left(\frac{\omega}{a}\right) = \frac{1}{|a|}\,\mathcal{F}\,u\left(\frac{\omega}{a}\right).$$ □

Note the corollary we obtain when $a = -1$.

Sign change

Corollary. $\mathcal{F}(u(-x))(\omega) = \mathcal{F}(u(x))(-\omega)$.

However, note also the following property.

Real symmetry

Theorem. If u is real-valued, then $\mathcal{F}\,u(-\omega) = \overline{\mathcal{F}\,u(\omega)}$.

Proof. Since $u(x) \in \mathbf{R}$, we have

$$\begin{aligned}\mathcal{F}\,u(-\omega) &= \int_{-\infty}^{\infty} u(x)e^{-i(-\omega)x}\,dx = \int_{-\infty}^{\infty} u(x)e^{i\omega x}\,dx = \int_{-\infty}^{\infty} \overline{u(x)e^{-i\omega x}}\,dx \\ &= \overline{\int_{-\infty}^{\infty} u(x)e^{-i\omega x}\,dx} = \overline{\mathcal{F}\,u(\omega)}.\end{aligned}$$

□

Note that this implies that if u is real valued and $U(\omega) = \mathcal{F}\,u(\omega)$, then

$$|U(-\omega)| = |U(\omega)|, \qquad \operatorname{Re} U(-\omega) = \operatorname{Re} U(\omega), \quad \text{and} \quad \operatorname{Im} U(-\omega) = -\operatorname{Im} U(\omega).$$

This means that there's symmetry around the imaginary axis for the spectrum of u. Remember though that this is only true in general when u is real-valued!

Translation

Theorem. Suppose that $a \in \mathbf{R}$ is constant. Then $\mathcal{F}(u(x-a))(\omega) = e^{-i\omega a}(\mathcal{F}(u(x)))(\omega)$.

Proof. A simple substitution shows that

$$\begin{aligned}\mathcal{F}(u(x-a))(\omega) &= \int_{-\infty}^{\infty} u(x-a)e^{-i\omega x}\,dx = /\ y = x-a\ / = \int_{-\infty}^{\infty} u(y)e^{-i\omega(y+a)}\,dy \\ &= e^{-i\omega a}\int_{-\infty}^{\infty} u(y)e^{-i\omega y}\,dy = e^{-i\omega a}\,\mathcal{F}\,u(\omega).\end{aligned}$$

□

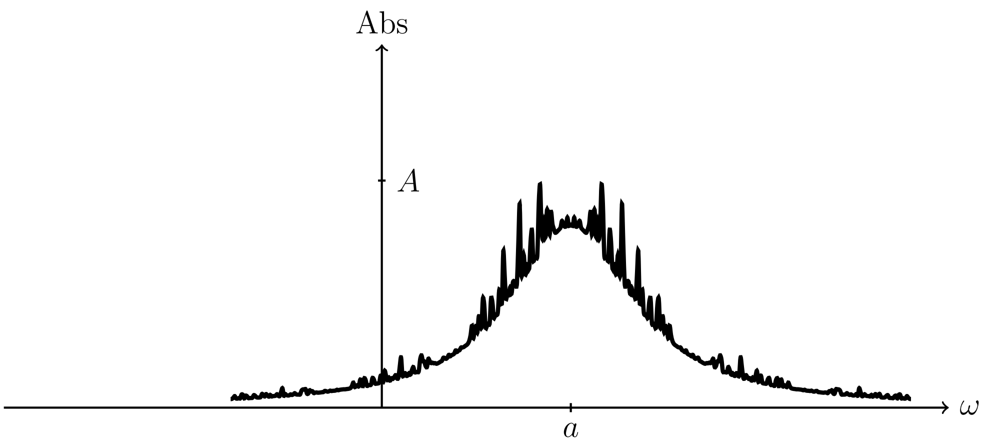
Phase shift

Theorem. Suppose that $a \in \mathbf{R}$ is constant. Then $\mathcal{F}(e^{iax}u(x))(\omega) = (\mathcal{F}(u(x)))(\omega - a)$.

Proof. We note that

$$\mathcal{F}(e^{iax}u(x))(\omega) = \int_{-\infty}^{\infty} u(x)e^{iax}e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} u(x)e^{-i(\omega-a)x}\,dx = \mathcal{F}\,u(\omega - a),$$

which completes the proof. □



Euler's formulas implies the following variation (that's useful in telecommunication).

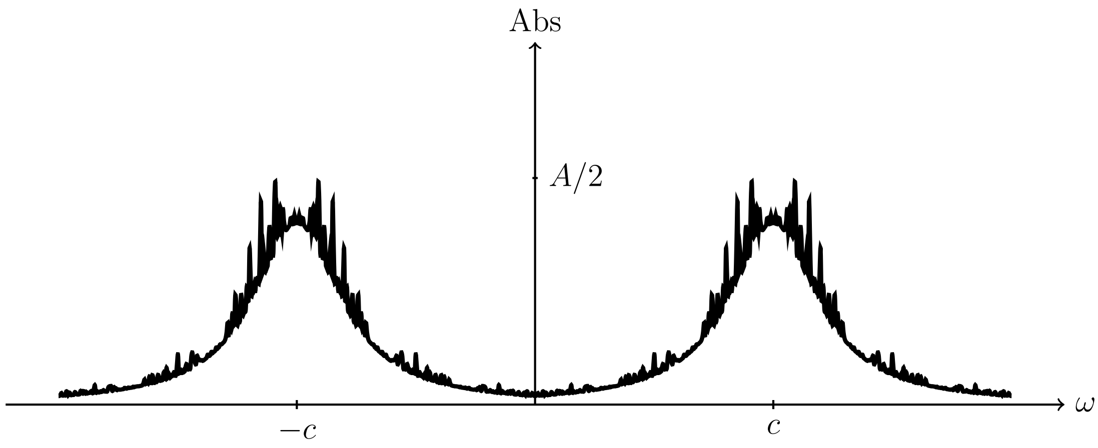
Modulation

Theorem. Suppose that $c \in \mathbf{R}$ is constant. Then

$$\mathcal{F}\big(u(x)\cos cx\big)(\omega) = \frac{\mathcal{F}(u(x))(\omega - c) + \mathcal{F}(u(x))(\omega + c)}{2}$$

and

$$\mathcal{F}\big(u(x)\sin cx\big)(\omega) = \frac{\mathcal{F}(u(x))(\omega - c) - \mathcal{F}(u(x))(\omega + c)}{2i}.$$



Notice that this means that we might get an overlap which might cause distortion in applications if the shift c is too small (if we just want a "copy" of the spectrum shifted to a higher frequency).

Complex conjugation

Theorem.

$$\mathcal{F}(\overline{u(x)})(\omega) = \overline{\mathcal{F}(u(x))(-\omega)}$$

Proof. By the linearity of the integral, clearly

$$\mathcal{F}(\overline{u(x)})(\omega) = \int_{-\infty}^{\infty} \overline{u(x)} e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} \overline{u(x)e^{-i(-\omega)x}}\,dx = \overline{\int_{-\infty}^{\infty} u(x)e^{-i(-\omega)x}\,dx} = \overline{\mathcal{F}\,u(-\omega)}. \quad \square$$

5.1 Differentiation

So let's move on to a very useful property of the Fourier transform: derivatives in one domain corresponds to multiplication by ω (or x) in the other domain. Formally, the proof is simple enough, but we need to exchange to order of integration and differentiation which is a bit problematic. So we need some preliminary results for how to handle expressions of the form $x^n u(x)$. But first, let's investigate what the Fourier transform of u' is.

Theorem. Let $u \in G(\mathbf{R})$ be differentiable and let $u' \in G(\mathbf{R})$. Then $\mathcal{F}(u')(\omega) = i\omega\,\mathcal{F}\,u(\omega)$.

Proof. First, since u is continuous, we have

$$u(x) - u(0) = \int_0^x u'(t)\,dt.$$

Since $u' \in G(\mathbf{R})$, we know that u' is absolutely integrable, and therefore the limit

$$\lim_{x\to\infty} u(x) = u(0) + \int_0^\infty u'(t)\,dt$$

exists. Furthermore, since u is also absolutely integrable and continuous, the limit above *must* be zero (if not then u, being continuous and having a limit at ∞, can't be absolutely integrable). Similarly we must have $u(x) \to 0$ as $x \to -\infty$. Using integration by parts, we see that

$$\begin{aligned}\int_{-M}^{M} u'(x)e^{-i\omega x}\,dx = /\ \text{I.B.P.}\ / = u(M)e^{-i\omega M} - u(-M)e^{i\omega M} + i\omega \int_{-M}^{M} u(x)e^{-i\omega x}\,dx \\ \to i\omega \int_{-\infty}^{\infty} u(x)e^{-i\omega x}\,dx = i\omega\,\mathcal{F}u(\omega),\ \text{ as } M\to\infty,\end{aligned}$$

since $u(x) \to 0$ as $x \to \pm\infty$. □

Theorem. Let $u \in G(\mathbf{R})$ be such that $xu(x) \in G(\mathbf{R})$. Then $\mathcal{F}(xu(x))(\omega) = i\dfrac{d}{d\omega}\mathcal{F}(u(x))(\omega)$.

"Proof." Formally, the proof is rather simple. Indeed, just observing that

$$\begin{aligned}U'(\omega) &= \frac{d}{d\omega}\int_{-\infty}^{\infty} u(x)e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} u(x)\frac{d}{d\omega}e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} -ixu(x)e^{-i\omega x}\,dx \\ &= -i\,\mathcal{F}(xu(x))(\omega),\end{aligned}$$

seems to indicate that the statement is true. However, the operation of moving the differential operator inside the integral is far from trivial; see the last section of this lecture.

Example

Find the Fourier transform of the gaussian e^{-x^2}.

Solution. One way of approaching this is by observing that both e^{-x^2} and xe^{-x^2} belong to $G(\mathbf{R})$, so if $u(x) = e^{-x^2}$ and $U(\omega) = \mathcal{F}u(\omega)$, then

$$\begin{aligned}U'(\omega) &= -i\,\mathcal{F}(xe^{-x^2})(\omega) = -i\int_{-\infty}^{\infty} xe^{-x^2}e^{-i\omega x}\,dx \\ &= /\ \text{I.B.P.}\ / = -i\left[-\frac{1}{2}e^{-x^2}e^{-i\omega x}\right]_{-\infty}^{\infty} - i\int_{-\infty}^{\infty}\frac{-i\omega}{2}e^{-x^2}e^{-i\omega x}\,dx = -\frac{\omega}{2}U(\omega).\end{aligned}$$

So U must satisfy

$$U'(\omega) + \frac{\omega}{2}U(\omega) = 0 \quad\Leftrightarrow\quad \frac{d}{d\omega}\left(e^{\omega^2/4}U(\omega)\right) = 0 \quad\Leftrightarrow\quad U(\omega) = Ce^{-\omega^2/4}.$$

However, we can only have one Fourier transform so we need to find a value for C. It is clear that

$$U(0) = \int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

This is a standard integral and one can for example find its value through the following calculation:

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \iint_{\mathbf{R}^2} e^{-(x^2+y^2)}\,dx\,dy$$
$$= \int_0^{\infty}\int_0^{2\pi} re^{-r^2}\,d\theta\,dr = 2\pi\left[-\frac{1}{2}\,e^{-r^2/2}\right]_0^{\infty} = \pi.$$

Therefore $C = U(0) = \sqrt{\pi}$ and we have shown that

$$\mathcal{F}\left(e^{-x^2}\right)(\omega) = \sqrt{\pi}e^{-\omega^2/4}.$$

Example

Find a solution to

$$u''(x) + 3u'(x) + 2u(x) = \begin{cases} e^{2x}, & x < 0, \\ e^{-x}, & x \geq 0. \end{cases}$$

Before jumping into the solution, let's ponder something that's a little worrying. The right-hand side is continuous (verify this), but not differentiable (at zero). What type of solution u can we expect to find that has this behavior when plugged into the equation? This is not a situation we've seen in previous analysis courses.

Solution. First, note that if $a > 0$, then

$$\mathcal{F}(e^{ax}H(-x))(\omega) = \int_{-\infty}^{0} e^{ax}e^{-i\omega x}\,dx = \left[\frac{e^{(a-i\omega)x}}{a-i\omega}\right]_{-\infty}^{0} = \frac{1}{a-i\omega}.$$

Similarly, if $a > 0$, then

$$\mathcal{F}(e^{-ax}H(x))(\omega) = \int_{0}^{\infty} e^{-ax}e^{-i\omega x}\,dx = \left[-\frac{e^{-(a+i\omega)x}}{a+i\omega}\right]_{0}^{\infty} = \frac{1}{a+i\omega}.$$

Since we can express the right-hand side as $e^{2x}H(-x) + e^{-x}H(x)$, where H is the Heaviside function, we obtain (assuming that $u \in G(\mathbf{R})$ and noting that the right-hand side is also in $G(\mathbf{R})$),

$$(i\omega)^2U(\omega) + 3i\omega U(\omega) + 2U(\omega) = \frac{1}{2-i\omega} + \frac{1}{1+i\omega} \quad\Leftrightarrow\quad ((i\omega)^2 + 3i\omega + 2)U(\omega) = \frac{1}{2-i\omega} + \frac{1}{1+i\omega}.$$

Notice that $(i\omega)^2 + 3i\omega + 2 = (i\omega+1)(i\omega+2)$, so we're looking for whatever has the transform

$$U(\omega) = \frac{1}{(1+i\omega)(2+i\omega)(2-i\omega)} + \frac{1}{(1+i\omega)^2(2+i\omega)}$$
$$= /\text{ partial fractions }/ = \frac{-2/3}{1+i\omega} + \frac{1}{(1+i\omega)^2} + \frac{3/4}{2+i\omega} + \frac{1/12}{2-i\omega}.$$

From a table (or the calculation above) we know that

$$\mathcal{F}(e^{-ax}H(X)) = \frac{1}{a+i\omega},$$

so the first and third term yields

$$-\frac{2}{3}e^{-x}H(x) + \frac{3}{4}e^{-2x}H(x).$$

Similarly, the last term yields

$$\frac{1}{12}e^{2x}H(-x).$$

To attack the remaining term, observe that

$$\frac{d}{d\omega}\left(\frac{1}{1+i\omega}\right) = -i\frac{1}{(1+i\omega)^2},$$

so since $\mathcal{F}(xu(x))(\omega) = iU'(\omega)$ (assuming that u is nice enough),

$$\mathcal{F}(xe^{-x}H(x))(\omega) = -i^2\frac{1}{(1+i\omega)^2} = \frac{1}{(1+i\omega)^2}.$$

Hence

$$\left(x - \frac{2}{3}\right)e^{-x}H(x) + \frac{3}{4}e^{-2x}H(x) + \frac{1}{12}e^{2x}H(-x)$$

has the Fourier transform $U(\omega)$. Therefore we suggest that

$$u(x) = \begin{cases} \frac{1}{12}e^{2x}, & x < 0, \\ \frac{3}{4}e^{-2x} + \left(x - \frac{2}{3}\right)e^{-x}, & x \geq 0, \end{cases}$$

is a solution to the differential equation. Directly verifying this proves the statement (something that should be done at this point). Even if we knew some uniqueness results, it is not completely easy to argue why this is a solution. One immediate concern is whether u is even differentiable at the origin!

Another reasonable question is why we only found one alternative. Shouldn't there be an infinite number of solutions considering the solutions to the homogeneous equation?

6 Principal Values and Integration

We're dealing with a lot of integrals in this part of the course, and this is not without rather deep issues. It is unfortunate that a lot of these issues stem from the fact that we're stuck with the Riemann integral, but that's as it may be. The price of moving over to another integral is rather high as well.

So, how do we interpret the Fourier transform? Considering that we've only considered absolutely integrable functions, things have behaved very nicely. But similarly with the case when summing Fourier series, we need to take into account the "phase" of the integrand sooner or later (in particular next lecture when introducing the inverse Fourier transform). To this end we introduce the principal value.

Definition. The **principal value** of an integral $\int_{-\infty}^{\infty} u(x)\,dx$ is defined as

$$\text{P. V.} \int_{-\infty}^{\infty} u(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} u(x)\,dx$$

whenever this limit exists.

Note here that the limit is of the symmetric summation of $u(x)$ around the origin. This in contrast with how we've introduced generalized integrals earlier, where the lower and upper limits were different and independent. This change will cause a larger class of functions to yield convergent integrals. To see that we're not messing up previous results, we show that if u is absolutely integrable, then

$$\text{P. V.} \int_{-\infty}^{\infty} u(x)\,dx = \int_{-\infty}^{\infty} u(x)\,dx.$$

In other words, the equality

$$\lim_{R\to\infty} \int_{-R}^{R} u(x)\,dx = \lim_{m\to-\infty} \int_{m}^{0} u(x)\,dx + \lim_{M\to\infty} \int_{0}^{M} u(x)\,dx$$

holds and is finite if $u \in L^1(\mathbf{R})$. This is clear since

$$\begin{aligned}\left| \int_{-\infty}^{\infty} u(x)\,dx - \int_{-R}^{R} u(x)\,dx \right| &= \left| \int_{-\infty}^{-R} u(x)\,dx + \int_{R}^{\infty} u(x)\,dx \right| \\ &\leq \int_{-\infty}^{-R} |u(x)|\,dx + \int_{R}^{\infty} |u(x)|\,dx \to 0,\end{aligned}$$

as $R \to \infty$ since u is absolutely integrable, which implies that both integrals in the right-hand side tend to zero (independently of each other). So in the case where we have absolutely integrable functions, the principal value integral will be equal to the integral with separate limits towards the infinities.

Now, let $f\colon \mathbf{R}^2 \to \mathbf{C}$ be a function of two real variables.

Uniform convergence of a principal value integral

Definition. We say that $F(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ converges uniformly on I if the integral exists for every x and

$$\sup_{x\in I} \left| \int_{-R}^{R} f(x,y)\,dy - F(x) \right| \to 0, \text{ as } R \to \infty.$$

Another useful concept (that's true in a setting a lot more general than ours) is that of dominated convergence. In a sense this is a uniform convergence, and as the following theorem shows we can use this to obtain uniform convergence as defined above.

Dominated convergence

Theorem. Suppose that $f\colon \mathbf{R}^2 \to \mathbf{C}$ and that $F(x) = \displaystyle\int_{-\infty}^{\infty} f(x,y)\,dy$ exists for all x. If there exists an absolutely integrable function $g\colon \mathbf{R} \to \mathbf{R}$ such that $|f(x,y)| \leq g(y)$ for all $x, y \in \mathbf{R}$, then $\displaystyle\int_{-\infty}^{\infty} f(x,y)\,dy$ converges uniformly on $\mathbf{R}$.

Proof. Let $F_R(x) = \displaystyle\int_{-R}^{R} f(x,y)\,dy$, $R > 0$. Since $F(x)$ exists for every x, it is clear that

$$|F(x) - F_R(x)| = \left| \int_{-\infty}^{-R} f(x,y)\,dy + \int_{R}^{\infty} f(x,y)\,dy \right| \leq \int_{-\infty}^{-R} |f(x,y)|\,dy + \int_{R}^{\infty} |f(x,y)|\,dy.$$

Observe now that $|f(x,y)| \leq g(y)$ implies that

$$\int_{-\infty}^{-R} |f(x,y)|\,dy \leq \int_{-\infty}^{-R} g(y)\,dy \to 0,$$

as $R \to \infty$ independently of x (since we know that g is absolutely integrable). Obviously the analogous result holds for $\displaystyle\int_{R}^{\infty} |f(x,y)|\,dy$. This proves that

$$\sup_{x\in\mathbf{R}} |F(x) - F_R(x)| \leq \int_{-\infty}^{-R} g(y)\,dy + \int_{R}^{\infty} g(y)\,dy \to 0,$$

as $R \to \infty$, which is uniform convergence. $\square$

Theorem. Suppose that $f\colon \mathbf{R}^2 \to \mathbf{C}$ is continuous on $[c,d] \times [a,R]$. Then

(i) $F_R(x) = \displaystyle\int_{a}^{R} f(x,y)\,dy$ is continuous on $[c,d]$

(ii) and if in addition f is continuous on $[c,d] \times [a,\infty[$ and $F(x) = \displaystyle\int_{a}^{\infty} f(x,y)\,dy$ converges uniformly (on $[c,d]$), then F is continuous.

Proof. This result is dependent on the uniform continuity of f on the closed set $[c,d] \times [a,R]$ (a continuous function on a compact set is always uniformly continuous), meaning that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|(x,y) - (x_0,y_0)| < \delta \quad \Rightarrow \quad |f(x,y) - f(x_0,y_0)| < \epsilon.$$

Note that δ is independent of the points x, y, x_0, y_0 (this is the uniformity).

(i) So, let $\epsilon > 0$ be fixed and choose $\delta > 0$ such that $|f(x+h,y) - f(x,y)| < \dfrac{\epsilon}{R-a}$ when $|h| < \delta$. Then

$$\begin{aligned} |F_R(x+h) - F_R(x)| &= \left| \int_a^R \big(f(x+h,y) - f(x,y)\big)\,dy \right| \\ &\leq \int_a^R |f(x+h,y) - f(x,y)|\,dy < \frac{\epsilon}{R-a} \int_a^R dy = \epsilon, \end{aligned}$$

which proves that F_R is continuous.

(ii) Since F_R is continuous and $F_R \to F$ uniformly on the interval $[c,d]$, it follows that F is continuous on $[c,d]$. $\square$

Exchanging the order of integration (Fubini's Theorem)

Theorem. Suppose that $f\colon \mathbf{R}^2 \to \mathbf{C}$ is a continuous function on $[c,d] \times [a,\infty[$ and that $F(x) = \int_a^\infty f(x,y)\,dy$ converges uniformly (on $[c,d]$). Then

$$\int_c^d \left(\int_a^\infty f(x,y)\,dy \right) dx = \int_a^\infty \left(\int_c^d f(x,y)\,dx \right) dy. \tag{6}$$

Proof. From standard multivariate analysis, we know that

$$\int_c^d \left(\int_a^R f(x,y)\,dy \right) dx = \int_a^R \left(\int_c^d f(x,y)\,dx \right) dy$$

for any constant $R > 0$. Now, by the uniform convergence, it is clear that

$$\begin{aligned}\int_a^\infty \left(\int_c^d f(x,y)\,dx \right) dy &= \lim_{R\to\infty} \int_a^R \left(\int_c^d f(x,y)\,dx \right) dy = \lim_{R\to\infty} \int_c^d \left(\int_a^R f(x,y)\,dy \right) dx \\ &= \lim_{R\to\infty} \int_c^d F_R(x)\,dx = \int_c^d \lim_{R\to\infty} F_R(x)\,dx = \int_c^d F(x)\,dx,\end{aligned}$$

which implies that (6) holds. $\square$

Note that we can let $a = -\infty$ in the previous theorems by exchanging $[a,R]$ by $[-R,R]$ and consider the principal values.

Leibniz rule

Theorem. Let $f\colon \mathbf{R}^2 \to \mathbf{C}$ be continuous and let $f'_x(x,y)$ exist and also be continuous. Suppose that $\int_{-\infty}^\infty f(x,y)\,dy$ is convergent for every x and suppose that $\int_{-\infty}^\infty f'_x(x,y)\,dy$ is uniformly convergent. Then

$$F'(x) = \frac{d}{dx} \int_{-\infty}^\infty f(x,y)\,dy = \int_{-\infty}^\infty f'_x(x,y)\,dy.$$

Proof. Let $G(x) = \int_a^\infty f'_x(x,y)\,dy$. Since this integral is assumed to be uniformly convergent and f'_x is continuous, it is clear that also G is continuous. Hence, for any $b \in \mathbf{R}$,

$$\begin{aligned}\int_b^x G(t)\,dt &= \int_b^x \int_{-\infty}^\infty f'_t(t,y)\,dy\,dt = \int_{-\infty}^\infty \int_b^x f'_t(t,y)\,dt\,dy \\ &= \int_{-\infty}^\infty \big(f(x,y) - f(b,y) \big)\,dy = F(x) - F(b).\end{aligned}$$

The fact that G is continuous proves that

$$\frac{d}{dx} \int_b^x G(t)\,dt = G(x),$$

so

$$F'(x) = \frac{d}{dx}\big(F(x) - F(b)\big) = G(x) = \int_{-\infty}^{\infty} f'_x(x,y)\,dy,$$

which is precisely want we wanted to show. □

7 Proof that $\mathcal{F}\,(xu(x))(\omega) = i(\mathcal{F}\,u(\omega))'$

The assumption was that $u \in G(\mathbf{R})$ and that $xu(x)$ is absolutely integrable (well.. we assumed that this product also belonged to $G(\mathbf{R})$ but given that $u \in G(\mathbf{R})$ this is equivalent). First, let us assume that u is continuous. Since $|e^{i\omega x}| = 1$ for $\omega \in \mathbf{R}$, it follows that the integral $\mathcal{F}(xu(x))(\omega)$ converges uniformly. By Leibniz' theorem, we can thus move the differentiation inside the integral obtaining that

$$\frac{d}{d\omega}\,\mathcal{F}(u)(\omega) = \int_{-\infty}^{\infty} u(x)\frac{d}{d\omega}e^{-i\omega x}\,dx = -i\int_{-\infty}^{\infty} xu(x)e^{-i\omega x}\,dx = -i\,\mathcal{F}(xu(x))(\omega),$$

which proves the claim in the case when u is continuous. If u has points of discontinuity, say $\{a_n\}_{n\in\mathbf{Z}}$ in increasing order, then the series

$$\mathcal{F}(xu(x))(\omega) = \sum_{n\in\mathbf{Z}} \int_{a_n}^{a_{n+1}} xu(x)e^{-i\omega x}\,dx$$

will converge uniformly, so by the argument above,

$$\frac{d}{d\omega}\,\mathcal{F}(u)(\omega) = \sum_{n\in\mathbf{Z}} \frac{d}{d\omega}\int_{a_n}^{a_{n+1}} u(x)e^{-i\omega x}\,dx = -i\sum_{n\in\mathbf{Z}} \int_{a_n}^{a_{n+1}} xu(x)e^{-i\omega x}\,dx = -i\,\mathcal{F}(xu(x))(\omega).$$

Note that $xu(x)$ will have at most the same discontinuity points as u.