Lecture 7: Inversion, Plancherel and Convolution

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"You should not drink and bake"
—Mark Kaminski

1 Inversion of the Fourier Transform

So suppose that we have $u \in G(\mathbf{R})$ and have calculated the Fourier transform $\mathcal{F}\,u(\omega)$. Can we from $\mathcal{F}\,u(\omega)$ recover the function we started with? Considering that the Fourier transform is constructed by the multiplication with $e^{-i\omega x}$ and then integration, what would happen if we multiplied with $e^{i\omega x}$ and integrate again? Formally,

$$\int_{-\infty}^{\infty} \mathcal{F}\,u(\omega)e^{i\omega x}\,d\omega = \lim_{R\to\infty}\int_{-R}^{R}\int_{-\infty}^{\infty} u(t)e^{-i\omega t}e^{i\omega x}\,dt\,d\omega = \lim_{R\to\infty}\int_{-R}^{R}\int_{-\infty}^{\infty} u(t)e^{-i\omega(x-t)}\,dt\,d\omega$$
$$= \lim_{R\to\infty}\int_{-\infty}^{\infty} u(t)\left(\int_{-R}^{R} e^{-i\omega(x-t)}\,d\omega\right)dt,$$

where we changed the order of integration (this can be motivated) but we're left with something kind of weird in the inner parenthesis and we would probably like to move the limit inside the outer integral. First, let's look at the expression in the inner parenthesis:

$$\int_{-R}^{R} e^{-i\omega(x-t)}\,d\omega = \left[\frac{e^{-i\omega(x-t)}}{-i(x-t)}\right]_{\omega=-R}^{R} = -\frac{e^{-iR(x-t)}}{i(x-t)} + \frac{e^{iR(x-t)}}{i(x-t)} = \frac{2\sin(R(x-t))}{x-t}, \quad x \neq t.$$

The Dirichlet kernel (on the real line)

Definition. We define the **Dirichlet kernel** for the Fourier transform by

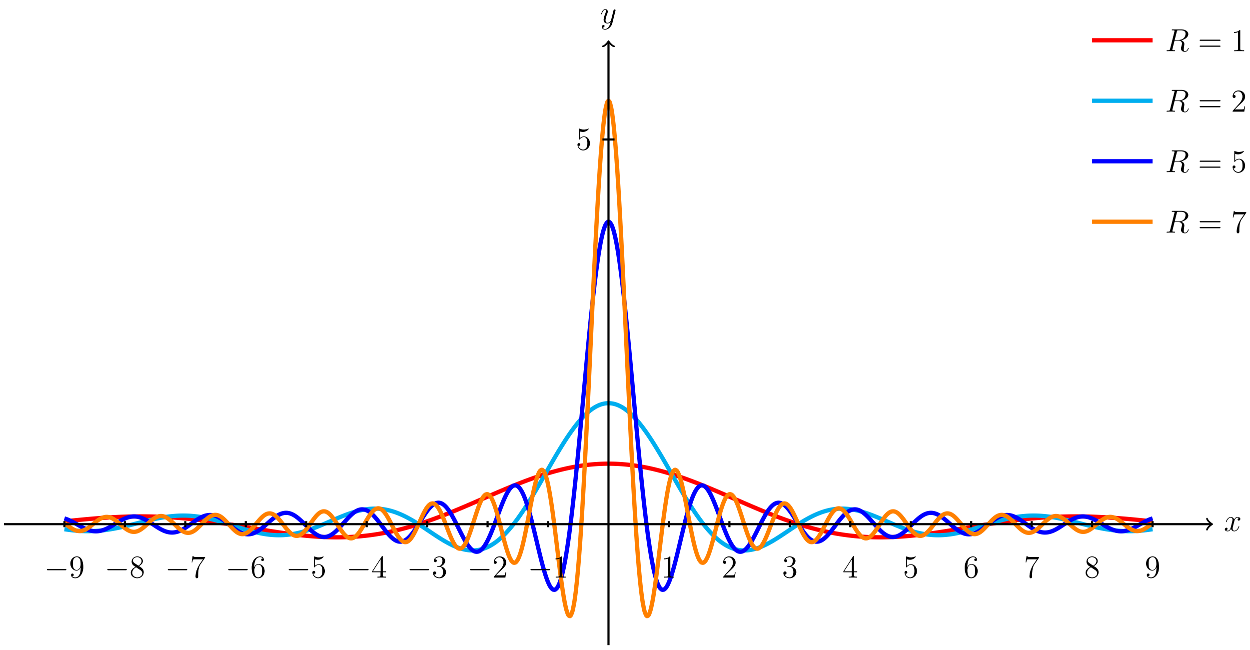
$$D_R(x) = \frac{\sin(Rx)}{\pi x}, \quad x \neq 0,\ R > 0,$$

and $D_R(0) = R/\pi$.

Note that we changed the *normalization* of the function. There's a reason for this and we'll get to that soon. For now, observe that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}\,u(\omega)e^{i\omega x}\,d\omega = \lim_{R\to\infty}\int_{-\infty}^{\infty} u(t)D_R(x-t)\,dt = \lim_{R\to\infty}\int_{-\infty}^{\infty} u(t+x)D_R(t)\,dt.$$

You probably recall the sinc-function, and the Dirichlet kernel on the real line is such a function and for a couple of values of R you can see the graphs below.



The Fourier inversion formula

Theorem. If $u \in G(\mathbf{R})$ has right- and lefthand derivatives at x, then

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \mathcal{F}\, u(\omega) e^{i\omega x}\, d\omega = \frac{u(x^+) + u(x^-)}{2}.$$

Proof. First, we write

$$\frac{1}{2\pi} \int_{-R}^{R} \mathcal{F}\, u(\omega) e^{i\omega x}\, d\omega = \int_{-\infty}^{0} u(t+x) D_R(t)\, dt + \int_{0}^{\infty} u(t+x) D_R(t)\, dt$$

and claim that

$$\int_{-\infty}^{0} u(t+x) D_R(t)\, dt \to \frac{u(x^-)}{2} \quad \text{and} \quad \int_{0}^{\infty} u(t+x) D_R(t)\, dt \to \frac{u(x^+)}{2},$$

as $R \to \infty$. We prove the second identity (the first is proved analogously). To this end, we split the integral in two parts:

$$\int_{0}^{\infty} u(t+x) D_R(t)\, dt = \int_{0}^{\pi} u(t+x) D_R(t)\, dt + \int_{\pi}^{\infty} u(t+x) D_R(t)\, dt.$$

The reason for this is that we need to exploit different properties of u to prove the desired result. First, let x be fixed. Then the function $t \mapsto t^{-1} u(t+x)$ is in $G(]\pi, \infty))$, so the Riemann Lebesgue lemma implies that

$$\lim_{R\to\infty} \int_{\pi}^{\infty} u(t+x) D_R(t)\, dt = \lim_{R\to\infty} \frac{1}{\pi} \int_{\pi}^{\infty} \frac{u(t+x)}{t} \sin(Rt)\, dt = 0.$$

Turning our attention to the first integral, we write

$$\begin{aligned} \int_{0}^{\pi} u(t+x) D_R(t)\, dt &= \int_{0}^{\pi} \big(u(t+x) - u(x^+)\big) D_R(t)\, dt + \int_{0}^{\pi} u(x^+) D_R(t)\, dt \\ &= \int_{0}^{\pi} \big(u(t+x) - u(x^+)\big) D_R(t)\, dt + u(x^+) \int_{0}^{\pi} D_R(t)\, dt. \end{aligned}$$

Since $D^+u(x)$ exists (by assumption), it is clear that the difference quotient

$$\frac{u(t+x)-u(x^+)}{t}$$

is bounded and that this expression belongs to $E([0,\pi])$. Therefore, the Riemann Lebesgue lemma (again!) implies that

$$\lim_{R\to\infty}\int_0^\pi \big(u(t+x)-u(x^+)\big)D_R(t)\,dt = \lim_{R\to\infty}\frac{1}{\pi}\int_0^\pi \frac{u(t+x)-u(x^+)}{t}\sin(Rt)\,dt = 0.$$

Finally, we observe that

$$\int_0^\pi D_R(t)\,dt = \big/\ x = Rt\ \big/ = \frac{1}{\pi R}\int_0^{R\pi}\frac{\sin x}{x/R}\,dx = \frac{1}{\pi}\int_0^{R\pi}\frac{\sin x}{x}\,dx \to \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2},\ \text{as } R\to\infty,$$

due to the following result.

Theorem. $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$

We defer the proof of this until at the end of the lecture.

Uniqueness

Corollary. If $u, v \in G(\mathbf{R})$ and $\mathcal{F}u(\omega) = \mathcal{F}v(\omega)$ for every $\omega \in \mathbf{R}$, then $u(x) = v(x)$ for all $x \in \mathbf{R}$ where u and v are continuous and $D^\pm u(x)$ and $D^\pm v(x)$ exists.

An Airy equation

Find a (formal) expression for a nonzero solution to $u''(x) - xu(x) = 0$.

Solution. Assuming that $u \in G(\mathbf{R})$ is twice differentiable with $u', u'' \in G(\mathbf{R})$, we can take the Fourier transform and obtain that

$$\begin{aligned}(i\omega)^2U(\omega) - iU'(\omega) = 0 \quad &\Leftrightarrow \quad U'(\omega) - i\omega^2 U(\omega) = 0\\ &\Leftrightarrow \quad \frac{d}{d\omega}\Big(e^{-i\omega^3/3}U(\omega)\Big) = 0 \quad \Leftrightarrow \quad U(\omega) = Ce^{i\omega^3/3},\end{aligned}$$

where C is an arbitrary constant (and we used an integrating factor to solve the differential equation). Therefore,

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Ce^{i\omega^3/3}e^{i\omega x}\,d\omega = D\int_{-\infty}^{\infty} e^{i(\omega^3/3+\omega x)}\,d\omega,$$

where D is some constant, might be an expression for a solution. Now the question is of course if this integral is convergent. Certainly it is not absolutely integrable (why?) and we can't claim that the expression solves the equation by previous results. This is an instance where we would like to extend the Fourier transform to a larger class of functions.

2 The Fourier Transform of the Fourier Transform

So looking at the inverse Fourier transform, it's almost the same as the Fourier transform. Indeed, the only difference is the sign in the exponent of the exponential and the factor before the integral. This means that the inverse transform has pretty much the same properties as the Fourier transform. This also means the following useful result.

Theorem. If $u, U \in G(\mathbf{R})$ and $U(\omega) = \mathcal{F}(u)(\omega)$, then

$$\mathcal{F}^{-1}(U)(x) = \frac{1}{2\pi}\mathcal{F}\big((\mathcal{F}\,u)(-\omega)\big)(x) \quad \text{and} \quad \mathcal{F}(\mathcal{F}\,u(\omega))(x) = 2\pi u(-x),$$

for every x where u is continuous and $D^{\pm}u(x)$ exist.

This follows immediately from the definitions of the transforms and the result above. The assumption that $D^{\pm}u(x)$ exist is superfluous but we do not know that at this point (we'll show this next lecture). If u is discontinuous, but still in $G(\mathbf{R})$, then the equalities still hold if we view the results as elements from $L^1(\mathbf{R})$, meaning that the difference has L^1-norm zero.

Example

Find the Fourier transform of $\dfrac{1}{1+x^2}$.

Solution. Let $u = \frac{1}{2}e^{-|x|}$. We know from before that $\mathcal{F}(u) = \mathcal{F}(e^{-|x|}/2)(\omega) = \dfrac{1}{1+\omega^2}$, and since both u and $x \mapsto \dfrac{1}{1+x^2}$ belong to $G(\mathbf{R})$ and are continuous with right- and lefthand derivatives at every point, we find that

$$2\pi u(-x) = \mathcal{F}(\mathcal{F}\,u(\omega))(x) = \mathcal{F}\left(\frac{1}{1+\omega^2}\right)(x),$$

so since $2\pi u(-x) = 2\pi \cdot \frac{1}{2}e^{-|-x|} = 2\pi \cdot \frac{1}{2}e^{-|x|}$, it is clear that $\mathcal{F}\left(\dfrac{1}{1+x^2}\right)(\omega) = \pi e^{-|\omega|}$.

3 Convolution

A useful type of "product" of two functions is the convolution (*sv. faltning*), defined as follows.

Convolution

Definition. The convolution $u * v\colon \mathbf{R} \to \mathbf{C}$ of two functions $u\colon \mathbf{R} \to \mathbf{C}$ and $v\colon \mathbf{R} \to \mathbf{C}$ is defined by

$$(u * v)(x) = \int_{-\infty}^{\infty} u(t)v(x-t)\,dt, \quad x \in \mathbf{R},$$

whenever this integral exists.

So when *does* this integral exist?

Theorem. If $u, v \in L^1(\mathbf{R})$, then $u * v \in L^1(\mathbf{R})$.

Proof. We first prove that $u * v$ is absolutely integrable:

$$\begin{aligned}
\int_{-\infty}^{\infty} |u * v(x)|\, dx = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} u(t)v(x-t)\, dt \right| dx \\
&\leq \text{/ monotonicity /} \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(t)v(x-t)|\, dt\, dx \\
&= \text{/ Fubini /} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(t)v(x-t)|\, dx\, dt \\
&= \int_{-\infty}^{\infty} |u(t)| \int_{-\infty}^{\infty} |v(x-t)|\, dx\, dt.
\end{aligned}$$

Note now that

$$\int_{-\infty}^{\infty} |v(x-t)|\, dx = \text{/ } s = x - t \text{ /} = \int_{-\infty}^{\infty} |v(s)|\, ds,$$

so

$$\int_{-\infty}^{\infty} |u(t)| \int_{-\infty}^{\infty} |v(x-t)|\, dx\, dt = \left(\int_{-\infty}^{\infty} |u(t)|\, dt \right) \left(\int_{-\infty}^{\infty} |v(s)|\, ds \right) < \infty.$$

A more compact way of stating this result is that

$$\|u * v\|_{L^1(\mathbf{R})} \leq \|u\|_{L^1(\mathbf{R})} \|v\|_{L^1(\mathbf{R})}.$$

The right-hand side is finite by assumption. This does ensure that $u * v$ exists as an element in $L^1(\mathbf{R})$, but we need to be a bit more precise in this course. $\square$

Theorem. If $u, v \in G(\mathbf{R})$ and either u or v is bounded, then $u * v \in L^1(\mathbf{R})$ is continuous and bounded.

Proof. Assume that u is bounded. Note that

$$|u * v(x)| = \left| \int_{-\infty}^{\infty} u(t)v(x-t)\, dt \right| \leq \|u\|_\infty \int_{-\infty}^{\infty} |v(x-t)|\, dt = \|u\|_\infty \|v\|_{L^1(\mathbf{R})},$$

so clearly $u * v$ is bounded. Moreover, using the same type of estimate,

$$|u * v(x+h) - u * v(x)| \leq \|u\|_\infty \underbrace{\int_{-\infty}^{\infty} |v(x+h-t) - v(x-t)|\, dt}_{\to 0 \text{ as } h \to 0}, \tag{1}$$

which would prove that $u * v$ is continuous. The fact that the second integral tends to zero however, is not obvious. So we need to show this. To this end, choose a continuous function w such that $w(x) = 0$ if $|x| > M$ and $\|v - w\|_{L^1(\mathbf{R})} < \epsilon/3$. This is possible due to the result in Section 6. Then (allowing some slight abuse of the notation)

$$\begin{aligned}
\|v(x+h-t) - v(x-t)\|_1 \leq \|v(x+h-t) - w(x+h-t)\|_1 + \|w(x+h-t) - w(x-t)\|_1 \\
+ \|w(x-t) - v(x-t)\|_1 \\
< \frac{\epsilon}{3} + \|w(x-t+h) - w(x-t)\|_1 + \frac{\epsilon}{3}.
\end{aligned}$$

We now use the fact that w is uniformly continuous on $[-M, M]$ (a continuous function on a compact set is always uniformly continuous), meaning that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$y \in [-M, M],\ |h| < \delta \quad \Rightarrow \quad |w(y+h) - w(y)| < \frac{\epsilon}{6M}.$$

Thus

$$\|w(x-t+h) - w(x-t)\|_1 = \int_{-\infty}^{\infty} \underbrace{|w(x-t+h) - w(x-t)|}_{< \frac{\epsilon}{6M} \text{ for all } x,t}\, dt \le \frac{2M\epsilon}{6M} = \frac{\epsilon}{3},$$

so if $|h|$ is small enough, then

$$\|v(x+h-t) - v(x-t)\|_1 < \epsilon,$$

which proves that the second integral in (1) tends to zero as $h \to 0$. $\square$

3.1 So What *Is* the Convolution?

The convolution is a type of moving average, where we shape one function by another. There are many (seriously.. there are a lot of them) applications where convolutions appear. Linear systems, (partial) differential equations, probability theory, integration theory, etc.

Example

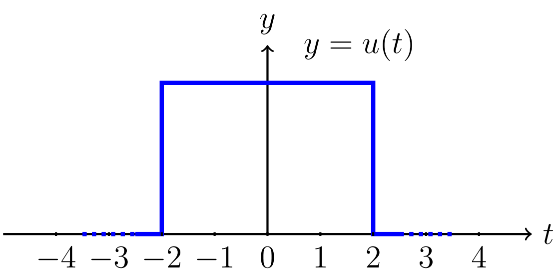
Let $u(x) = 5H(x+2) - 5H(x-2)$ and $v(x) = 4H(x+1) - 4H(x-2)$, that is,

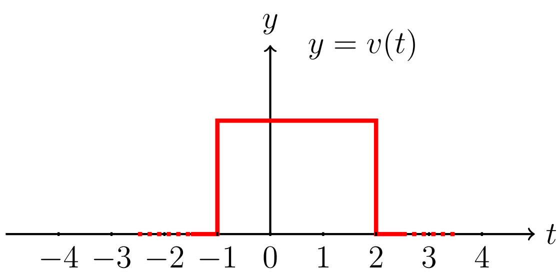
$$u(x) = \begin{cases} 5, & -2 \le x \le 2, \\ 0, & \text{elsewhere} \end{cases} \qquad \text{and} \qquad v(x) = \begin{cases} 4, & -1 \le x \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the convolution $u * v(x)$.

Solution. Since both functions are defined by cases, a reasonable procedure is as follows.

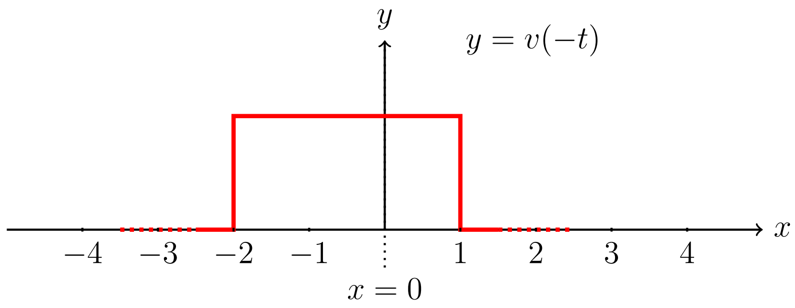
(i) First, identify where the functions have jumps (or where the *support*[1] is if it is compact). We also express both functions in terms of a variable t that's going to disappear when we integrate.





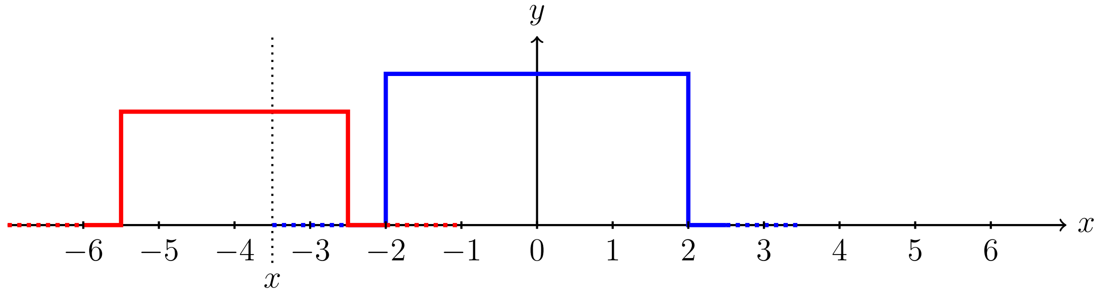
(ii) Now we mirror v, so lets draw $y = v(-t)$. Since we will consider $v(x-t)$, this graph corresponds to $x = 0$. We need to keep track of where x is.

[1]The support of a function $u\colon \mathbf{R} \to \mathbf{C}$ is the smallest closed set E such that $\{x \in \mathbf{R} : u(x) \neq 0\} \subset E$.



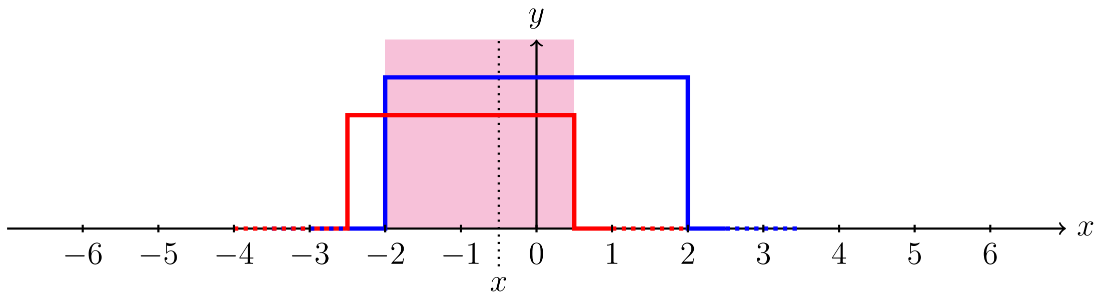
(iii) Draw both $u(t)$ and $v(x-t)$ in the same diagram, identifying when things change.

First, we see that for $x < -3$, we have no overlap.



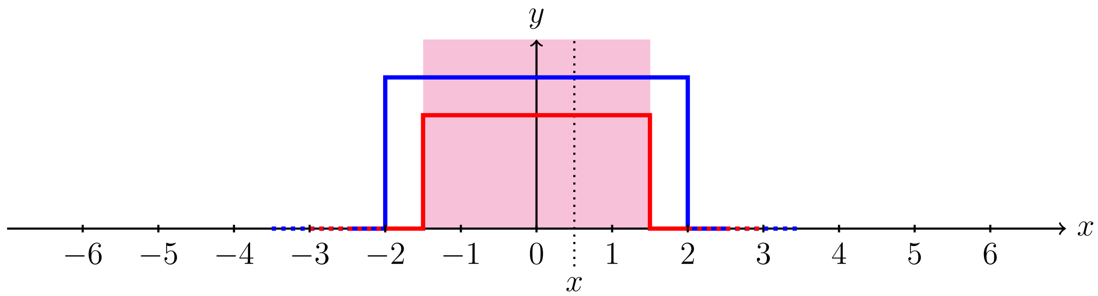
Obviously, $u * v(x) = 0$ for $x < -3$.

For $-3 \leq x \leq 0$, we have some overlap:



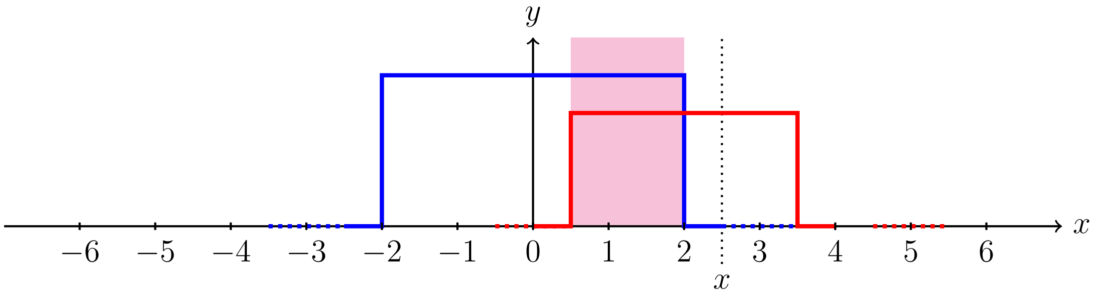
$$u * v(x) = \int_{-2}^{x+1} u(t)v(x-t)\,dt = \int_{-2}^{x+1} 5 \cdot 4\,dt = 20(x+3).$$

For $0 \leq x \leq 1$, we have complete overlap:



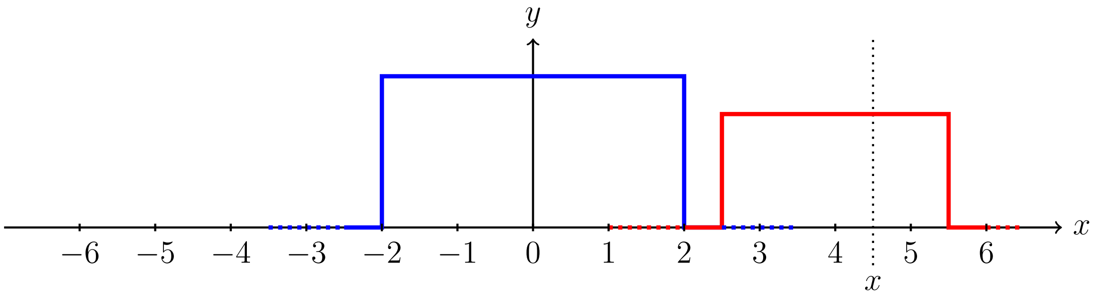
$$u * v(x) = \int_{x-2}^{x+1} u(t)v(x-t)\,dt = \int_{x-2}^{x+1} 5 \cdot 4\,dt = 20(x+1-x+2) = 20 \cdot 3.$$

For $1 \le x \le 4$, we have some overlap:



$$u * v(x) = \int_{x-2}^{2} u(t)v(x-t)\,dt = \int_{x-2}^{2} 5 \cdot 4\,dt = 20(4-x).$$

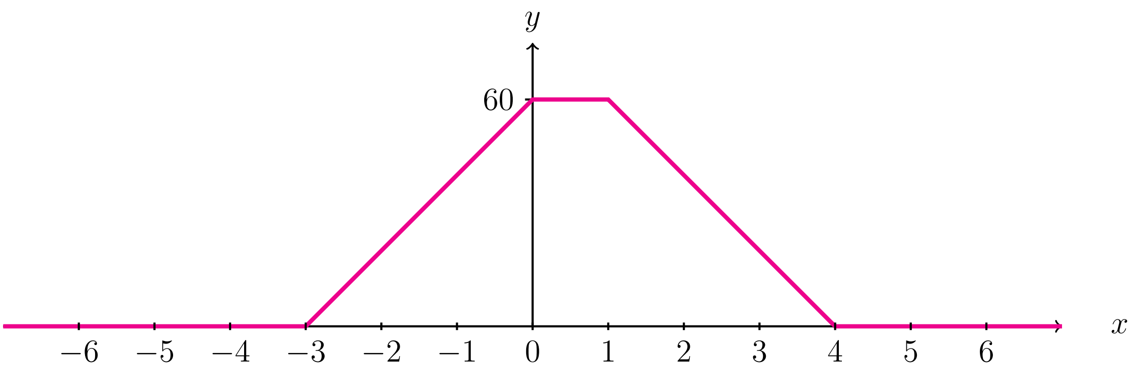
For $x > 4$, there is no overlap so $u * v(x) = 0$.



We have now covered all possibilities for x, so the answer is

$$u * v(x) = \begin{cases} 0, & x < -3, \\ 20(x+3), & -3 \le x < 0, \\ 60, & 0 \le x < 1, \\ 20(4-x), & 1 \le x \le 4, \\ 0, & x > 4, \end{cases}$$

and the graph looks like this.



Note that u and v are discontinuous, but the convolution $u * v$ is a continuous function. This is rather typical (forming the convolution is a smoothing operation).

3.2 The Fourier Transform

So now to one of the most important properties of the Fourier transform: the Fourier transform of the convolution of u and v is the product of the Fourier transforms of u and v (separately).

Convolution

Theorem. Suppose that $u, v \in G(\mathbf{R})$ with either function bounded. Then $\mathcal{F}(u * v)(\omega) = \mathcal{F}\,u(\omega)\,\mathcal{F}\,v(\omega)$.

Proof. Let $\mathcal{F}\,u(\omega) = U(\omega)$ and $\mathcal{F}\,v(\omega) = V(\omega)$. Then

$$\begin{aligned}\mathcal{F}(u * v)(\omega) &= \int_{-\infty}^{\infty} (u * v)(x) e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} u(t)v(x-t)\,dt \right) e^{-i\omega x}\,dx \\ &= /\text{ Fubini }/ = \int_{-\infty}^{\infty} u(t) \int_{-\infty}^{\infty} v(x-t)e^{-i\omega x}\,dx\,dt = \int_{-\infty}^{\infty} u(t)V(\omega)e^{-i\omega t}dt \\ &= V(\omega)\int_{-\infty}^{\infty} u(t)e^{-i\omega t}dt = V(\omega)U(\omega).\end{aligned}$$

Example

Find a solution to the integral equation

$$\int_{-\infty}^{\infty} u(t)u(x-t)\,dt = e^{-x^2}.$$

Solution. The left-hand side is the convolution of u with itself. Assume that $u \in G(\mathbf{R})$. Then taking the Fourier transform of both sides in the equality yields

$$U(\omega)U(\omega) = \mathcal{F}(e^{-x^2})(\omega) = \sqrt{\pi}e^{-\omega^2/4}$$

so assuming that U is real-valued (is this obvious?),

$$|U(\omega)| = \sqrt{\sqrt{\pi}e^{-\omega^2/4}} = \pi^{1/4}e^{-\omega^2/8},$$

so

$$U(\omega) = \pm\pi^{1/4}e^{-\omega^2/8} = \pm\pi^{-1/4}\sqrt{2}\cdot\frac{1}{\sqrt{2}}\sqrt{\pi}e^{-(\omega/\sqrt{2})^2/4} = \pm\pi^{-1/4}\sqrt{2}\cdot\frac{1}{\sqrt{2}}F\left(\frac{\omega}{\sqrt{2}}\right),$$

where we rewrote the right-hand side in term of $F(\omega) = \mathcal{F}(e^{-x^2})(\omega)$. Why? Because we want to use the formula $\mathcal{F}(u(ax))(\omega) = |a|^{-1}\,\mathcal{F}\,u(\omega/a)$. Hence

$$u(x) = \pm\pi^{-1/4}\sqrt{2}e^{-(x\sqrt{2})^2} = \pm\pi^{-1/4}\sqrt{2}e^{-2x^2},$$

by the scaling argument. Is this a solution? Yes, by uniqueness (obviously u is continuously differentiable and also bounded).

3.3 Properties of the Convolution Product

The convolution operation (on $L^1(\mathbf{R})$) behaves like we expect of a product in that it has the following properties.

Theorem. Suppose that $u, v, w \in G(\mathbf{R})$. Then the convolution has the following properties (assuming that at least one factor in each convolution is bounded).

(i) Associative: $(u * v) * w(x) = u * (v * w)(x)$.

(ii) Distributive: $(u + v) * w(x) = u * w(x) + v * w(x)$.

(iii) Commutative: $u * v(x) = v * u(x)$.

Proof. Since the convolution of functions from $G(\mathbf{R})$ are mapped to the product of their respective Fourier transforms, all of these properties follow from the fact that they hold for the regular product. Taking the Fourier transform of both sides of the equations (which is allowed since everything belongs to $G(\mathbf{R})$), we see that the identities hold for the Fourier transforms. We then need to use a fact that we will show on the next lecture: if $f, g \in G(\mathbf{R})$ are continuous at x, then $\mathcal{F} f(\omega) = \mathcal{F} v(\omega)$ (for all ω) implies that $f(x) = g(x)$. $\square$

Note that these properties are only guaranteed when the elements belong to $G(\mathbf{R})$. It is also quite possible to directly prove that these properties hold from the definition of the convolution.

An interesting question is if there is a unit for the convolution? That is, is there some element δ such that $u * \delta = u$ for all u? It turns out that this is *not* possible with $\delta \in L^1(\mathbf{R})$, but moving over to *distributions*, we can consider the Dirac impulse "function."

4 Plancherel's Formula

Recall that the space $L^2(\mathbf{R})$ consists of those functions $u \colon \mathbf{R} \to \mathbf{C}$ such that

$$\int_{-\infty}^{\infty} |u(x)|^2 \, dx < \infty.$$

It is true that if $u \in G(\mathbf{R}) \cap L^2(\mathbf{R})$, then $\mathcal{F} u \in L^2(\mathbf{R})$. This fact is far from trivial, but the following result holds.

Plancherel's theorem / Parseval's identity

Theorem. Suppose that $u \in G(\mathbf{R}) \cap L^2(\mathbf{R})$. Then

$$\int_{-\infty}^{\infty} |u(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{F} u(\omega)|^2 \, d\omega.$$

One can view this result through different lenses (this was true also for the corresponding result for Fourier series). You can choose to consider this just a formula that connects the energy (square-integrals are usually regarded as energy integrals when dealing with physics) of the function with that of its Fourier transform. On the other hand, you can view this as a deeper result where we see that the Fourier transform always maps an element of $L^1 \cap L^2$ into L^2 and does so in a bounded way (we have control of the L^2-norm of the transform in terms of the L^2-norm of the function we started with). This allows us to *extend* the Fourier transform to the whole class of functions in $L^2(\mathbf{R})$. The scope of this is outside reasonable limits for this course, so we'll just leave it at that.

Analogously with the case for Fourier series (using the polarization identity), we can obtain the following generalization.

Plancherel's (generalized) formula

Theorem. Suppose that $u, v \in G(\mathbf{R}) \cap L^2(\mathbf{R})$. Then

$$\int_{-\infty}^{\infty} u(x)\,\overline{v(x)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}\,u(\omega)\,\overline{\mathcal{F}\,v(\omega)}\,d\omega.$$

The proof of Plancherel's identity follows from Parseval's using the same polarization identity that was used for the corresponding proof for Fourier series. So we focus on proving Parseval's identity.

Proof. The first question is that it is not clear *a priori* that the integrals involved are defined. Remember that the Fourier transform $\mathcal{F}(u)$ is uniformly bounded by the L^1-norm of u, but what about the L^2-norm of $\mathcal{F}(u)$?

To attack this problem, we first assume that $u, v \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ are twice continuously differentiable (meaning of class $C^2(\mathbf{R})$) and have *compact support* (meaning basically that the functions are zero outside of a compact set, say $[-M, M]$ in our case). Then it is clear that $\mathcal{F}(u'')(\omega) = -\omega^2\,\mathcal{F}\,u(\omega)$, so

$$|\,\mathcal{F}\,u(\omega)| = \left|\frac{\mathcal{F}(u'')(\omega)}{\omega^2}\right| \le \frac{C}{|\omega|^2}, \quad \omega \neq 0,$$

where $C > 0$ exists due to the fact that we have the uniform bound

$$|\,\mathcal{F}(u'')(\omega)| \le \|u''\|_{L^1(\mathbf{R})} < \infty$$

and u'' is continuous and $u''(x) = 0$ for $|x| > M$ for some constant M. Since also $\mathcal{F}(u)$ is continuous, this implies that $\mathcal{F}(u) \in L^1(\mathbf{R})$ (and obviously also $L^2(\mathbf{R})$). Analogously, it follows that $\mathcal{F}\,v \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Moreover, $\mathcal{F}^{-1}\,\mathcal{F}\,v = v$ (recall that v is also continuous). Then

$$\begin{aligned}
\int_{-\infty}^{\infty} u(x)\overline{v(x)}\,dx &= \int_{-\infty}^{\infty} u(x)\overline{\mathcal{F}^{-1}(\mathcal{F}(v))(x)}\,dx \\
&= \int_{-\infty}^{\infty} u(x)\overline{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(v)(\omega)e^{i\omega x}\,d\omega}\,dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)\int_{-\infty}^{\infty} \overline{\mathcal{F}(v)(\omega)}e^{-i\omega x}\,d\omega\,dx \\
&= /\text{ Fubini } / = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} u(x)e^{-i\omega x}\,dx\right)\overline{\mathcal{F}(v)(\omega)}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(u)(\omega)\,\overline{\mathcal{F}(v)(\omega)}\,d\omega,
\end{aligned}$$

which implies Parseval's formula (for functions in $C^2(\mathbf{R}) \cap L^1(\mathbf{R}) \cap L^2(\mathbf{R})$).

So the next question becomes if we can somehow approximate — in some useful sense — a general function u by something in C^2. And the answer is yes, although we defer the proof until the end of the lecture. For any $\epsilon > 0$, there exists a function $v \in C^2(\mathbf{R}) \cap L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ such that

$$\|u - v\|_{L^1(\mathbf{R})} < \epsilon \qquad \text{and} \qquad \|u - v\|_{L^2(\mathbf{R})} < \epsilon.$$

This means that we can choose a sequence $v_1, v_2, v_3, \ldots$ such that $v_k \to u$ in both $L^1(\mathbf{R})$ and $L^2(\mathbf{R})$ (norm convergence). To simplify (without loss of generality as it turns out), we will only prove Parseval's identity (Plancherel's formula follows as stated previously). So let u belong to $G(\mathbf{R}) \cap L^2(\mathbf{R})$. First we prove that $\mathcal{F}\, u \in L^2(\mathbf{R})$. To this end, let $\Omega_R = [-R, R]$ and observe that

$$\|\mathcal{F}\, u\|_{L^2(\Omega_R)} \leq \|\mathcal{F}\, u - \mathcal{F}\, v_k\|_{L^2(\Omega_R)} + \|\mathcal{F}\, v_k\|_{L^2(\Omega_R)} \leq \|\mathcal{F}\, u - \mathcal{F}\, v_k\|_{L^2(\Omega_R)} + K, \tag{2}$$

where if A is a reasonable set (like a union of intervals),

$$\|w\|_{L^2(A)} := \left(\int_A |w(x)|^2 \, dx \right)^{1/2},$$

and $K > 0$ is some constant such that

$$K^2 \geq 2\pi \int_{-\infty}^{\infty} |v_k(x)|^2 \, dx = \int_{-\infty}^{\infty} |\mathcal{F}\, v_k(\omega)|^2 \, d\omega \geq \|\mathcal{F}\, v_k\|^2_{L^2(\Omega_R)}, \quad \text{for all } k = 1, 2, 3, \ldots.$$

This is possible since $\|v_k\|_{L^2(\mathbf{R})} \to \|u\|_{L^2(\mathbf{R})}$ by continuity, so the sequence of norms must be bounded. Indeed, the continuity of the norm is true in general: for any normed linear space X, the function $\|\cdot\| \colon X \to [0, \infty[$ is continuous due to the (reverse) triangle inequality:

$$|\|u\| - \|v\|| \leq \|u - v\|,$$

so for any $\epsilon > 0$, if $\|u - v\| < \delta = \epsilon$, then $|\|u\| - \|v\|| < \epsilon$. Obviously, this implies that also $\|\cdot\|^\alpha$ is continuous on X for any $\alpha > 0$.

Note also that K in (2) is independent of R. Now, since

$$\sup_{\omega \in \mathbf{R}} |\mathcal{F}(u - v_k)(\omega)| \leq \|u - v_k\|_{L^1(\mathbf{R})},$$

we obtain that

$$\begin{aligned} \|\mathcal{F}\, u - \mathcal{F}\, v_k\|_{L^2(\Omega_R)} &= \left(\int_{-R}^{R} |\mathcal{F}(u - v_k)(\omega)|^2 \, d\omega \right)^{1/2} \leq \left(\int_{-R}^{R} \|\mathcal{F}(u - v_k)\|^2_\infty \, d\omega \right)^{1/2} \\ &\leq \|u - v_k\|_{L^1(\mathbf{R})} \sqrt{2R} \to 0, \end{aligned} \tag{3}$$

as $k \to \infty$ for any $R > 0$. Letting $k \to \infty$ also completes the proof that $\mathcal{F}\, u \in L^2(\mathbf{R})$ since the bound is independent of R so we can let $R \to \infty$ after letting $k \to \infty$ (the order here is important).

We can now consider the following expression, where the integrals are convergent by the argument above. So, by the triangle inequality,

$$\begin{aligned} \left| 2\pi \int_{-\infty}^{\infty} |u(x)|^2 \, dx - \int_{-\infty}^{\infty} |\mathcal{F}\, u(\omega)|^2 \, d\omega \right| &\leq \left| 2\pi \int_{-\infty}^{\infty} |u(x)|^2 \, dx - 2\pi \int_{-\infty}^{\infty} |v_k(x)|^2 \, dx \right| \\ &\quad + \left| 2\pi \int_{-\infty}^{\infty} |v_k(x)|^2 \, dx - \int_{-\infty}^{\infty} |\mathcal{F}\, u(\omega)|^2 \, d\omega \right|. \end{aligned} \tag{4}$$

Note that

$$2\pi \int_{-\infty}^{\infty} |u(x)|^2 \, dx - 2\pi \int_{-\infty}^{\infty} |v_k(x)|^2 \, dx = 2\pi \left(\|u\|_2^2 - \|v_k\|_2^2 \right) \to 0, \text{ as } k \to \infty, \tag{5}$$

since $\|v_k\|_2 \to \|u\|_2$. Moreover, since $v_k \in C^2 \cap G(\mathbf{R}) \cap L^2(\mathbf{R})$, it is true that

$$2\pi \int_{-\infty}^{\infty} |v_k(x)|^2\, dx = \int_{-\infty}^{\infty} |\,\mathcal{F}\, v_k(\omega)|^2\, d\omega,$$

so

$$2\pi \int_{-\infty}^{\infty} |v_k(x)|^2\, dx - \int_{-\infty}^{\infty} |\,\mathcal{F}\, u(\omega)|^2\, d\omega = \|\,\mathcal{F}\, v_k\|_2^2 - \|\,\mathcal{F}\, u\|_2^2.$$

We want to show that $\|\,\mathcal{F}\, v_k\|_2 \to \|\,\mathcal{F}\, u\|_2$, and by the (reverse) triangle inequality we have

$$|\|\,\mathcal{F}\, v_k\|_2 - \|\,\mathcal{F}\, u\|_2| \le \|\,\mathcal{F}\, v_k - \mathcal{F}\, u\|_2 = \|\,\mathcal{F}(v_k - u)\|_2 = \left(\int_{-\infty}^{\infty} |\,\mathcal{F}(v_k - u)(\omega)|^2\, d\omega\right)^{1/2}.$$

Recalling that the Fourier transform maps $G(\mathbf{R})$-functions into uniformly bounded functions, it is true that

$$|\,\mathcal{F}(v_k - u)(\omega)| \le \int_{-\infty}^{\infty} |v_k - u|\, dx,$$

where the right-hand side tends to zero (uniformly in ω). To exploit this, we need to split the integral into two parts before letting $k \to \infty$. Note that $\|\,\mathcal{F}\, v_k\|_2 = \sqrt{2\pi}\|v_k\|_2 \to \sqrt{2\pi}\|u\|_2$ implies that there exists a number N such that

$$\|\,\mathcal{F}\, v_k - \mathcal{F}\, v_n\|_2 < \frac{\epsilon}{3}, \quad k, n \ge N.$$

Let $n \ge N$ be fixed and choose $R > 0$ such that

$$\int_{|\omega|>R} |\,\mathcal{F}\, v_n(\omega)|^2\, d\omega < \frac{\epsilon^2}{9} \qquad \text{and} \qquad \int_{|\omega|>R} |\,\mathcal{F}\, u(\omega)|^2\, d\omega < \frac{\epsilon^2}{9}. \tag{6}$$

This is possible since $\mathcal{F}\, v_k, \mathcal{F}\, u \in L^2(\mathbf{R})$. Now,

$$\|\,\mathcal{F}(v_k - u)\|_{L^2(\mathbf{R})} \le \|\,\mathcal{F}(v_k - u)\|_{L^2(\Omega_R)} + \|\,\mathcal{F}(v_k - u)\|_{L^2(\Omega_R^c)},$$

and for any $R > 0$, $\|\,\mathcal{F}(v_k - u)\|_{L^2(\Omega_R)} \to 0$ due to (3). Furthermore,

$$\begin{aligned}\|\,\mathcal{F}(v_k - u)\|_{L^2(\Omega_R^c)} &\le \|\,\mathcal{F}(v_k - v_n)\|_{L^2(\Omega_R^c)} + \|\,\mathcal{F}(v_n - u)\|_{L^2(\Omega_R^c)} \\ &\le \|\,\mathcal{F}(v_k - v_n)\|_{L^2(\mathbf{R})} + \|\,\mathcal{F}(v_n - u)\|_{L^2(\Omega_R^c)} < \frac{\epsilon}{3} + \|\,\mathcal{F}(v_n - u)\|_{L^2(\Omega_R^c)}\end{aligned}$$

and

$$\|\,\mathcal{F}(v_n - u)\|_{L^2(\Omega_R^c)} \le \|\,\mathcal{F}\, v_n\|_{L^2(\Omega_R^c)} + \|\,\mathcal{F}\, u\|_{L^2(\Omega_R^c)} < \frac{2\epsilon}{3}$$

because of (6). Hence

$$\|\,\mathcal{F}(v_k - u)\|_{L^2(\mathbf{R})} \le 2R\|v_k - u\|_{L^1(\mathbf{R})} + \epsilon.$$

Letting $k \to \infty$ we find that $\|\,\mathcal{F}(v_k - u)\|_2 < \epsilon$ and since $\epsilon > 0$ was arbitrary, this proves that $\|\,\mathcal{F}\, v_k\|_2 \to \|\,\mathcal{F}\, u\|_2$ as $k \to \infty$. This also completes the proof that the right-hand side of (4) can be made arbitrarily small. □

So that was a lengthy piece of mathematics, which is a bit unfortunate considering that the proof could be made very short if we just had a couple of new tools. The idea is using the fact that all elements in $L^2(\mathbf{R})$ can be approximated by smooth functions with compact support, so we can extend the Fourier transform by continuity due to the fact that Parseval's formula hold for this smaller set. This is basically what we do explicitly above, but extending by continuity would hide all of the messiness. This is a regular technique in functional analysis when working with linear operators (we define a bounded operator explicitly on a dense subspace and then extend the operator so that this bound still hold).

Nevertheless, the proof includes some nice mathematical arguments where it is important to keep track of the order we do certain approximations so try to go through it and see why the order is important.

So how will we use this result? Similarly to Parseval's formula in the case of Fourier series, we can find the value for certain generalized integrals in this manner.

Example

Calculate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2}\, d\omega$.

Solution. We observe that $U(\omega) = \dfrac{1}{1+\omega^2}$ is the Fourier transform of $u(x) = \dfrac{1}{2}\, e^{-|x|}$. Since it is clear that $u \in G(\mathbf{R}) \cap L^2(\mathbf{R})$, Plancherel's formula implies that

$$\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2}\, d\omega = 2\pi \int_{-\infty}^{\infty} \left(\frac{1}{2}\, e^{-|x|}\right)^2 dx = 4\pi \int_{0}^{\infty} \left(\frac{1}{2}\, e^{-|x|}\right)^2 dx = \pi \int_{0}^{\infty} e^{-2x}\, dx$$
$$= \pi \left[-\frac{e^{-2x}}{2}\right]_0^{\infty} = \frac{\pi}{2}.$$

5 Proof That $\displaystyle\int_0^{\infty} \operatorname{sinc}(x)\, dx = \frac{\pi}{2}$

First we prove that

$$\int_0^{\infty} \frac{\sin x}{x}\, dx \tag{7}$$

is convergent. The idea is that if we know this, we can choose a particular way for the upper limit to approach infinity (and be sure that this is the correct value).

Note that since $\sin(x)/x$ is bounded and continuous (the limit when $x \to 0$ is 1), it is clear that

$$\int_0^{\pi} \frac{\sin x}{x}\, dx$$

is convergent. Now, using integration by parts we obtain

$$\int_{\pi}^{b} x^{-1} \sin x\, dx = \left[-x^{-1}\cos x\right]_{\pi}^{b} - \int_{\pi}^{b} \frac{\cos x}{x^2}\, dx.$$

The integral in the right-hand side is absolutely convergent since

$$\int_{\pi}^{b} \left|\frac{\cos x}{x^2}\right| dx \leq \int_{\pi}^{b} \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_{\pi}^{b} = \frac{1}{\pi} - \frac{1}{b} \to \frac{1}{\pi}$$

as $b \to \infty$ (the exact number is not important and similarly the number π is arbitrary). So the conclusion is that (7) is convergent.

Since (7) is convergent, we can find it's value by the following calculation:

$$\begin{aligned}
\int_0^\infty \frac{\sin x}{x}\,dx &= \lim_{\mathbf{Z}\ni m\to\infty} \int_0^{(m+1/2)\pi} \frac{\sin x}{x}\,dx \\
&= \Big/\ t = \frac{x}{m+1/2}\ \Big/ = \lim_{\mathbf{Z}\ni m\to\infty} \int_0^{\pi} \frac{\sin(t(m+1/2))}{t}\,dt \\
&= \lim_{\mathbf{Z}\ni m\to\infty} \frac{1}{2}\int_0^{\pi} \frac{2\sin(t/2)}{t} D_m(t)\,dt
\end{aligned}$$

where $D_m(t)$ is the Dirichlet kernel on $[-\pi,\pi]$, that is

$$D_m(t) = \sum_{k=-m}^{m} e^{-ikt} = \frac{\sin(t(2m+1)/2)}{\sin(t/2)};$$

see Lecture 3. Moreover, the convergence result from Lecture 3 shows that if $u \in E'[-\pi,\pi]$, then

$$\lim_{m\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t+x) D_m(t)\,dt = \frac{u(x^+)+u(x^-)}{2}.$$

So letting $u(t) = \dfrac{2\sin(t/2)}{t}$ for $0 \le t \le \pi$ and $u = 0$ for $-\pi < t < 0$ (and extended periodically), we observe that obviously $u \in E$ and we see that

$$\begin{aligned}
D^+u(0) &= \lim_{h\to 0^+} \frac{u(h)-u(0)}{h} = \lim_{h\to 0^+} \frac{2\sin(h/2)/h - 1}{h} = \lim_{h\to 0^+} \frac{1}{h^2}\left(2\sin(h/2) - h\right) \\
&= \lim_{h\to 0^+} \frac{1}{h^2}\Big(2\left(h/2 + O(h^3)\right) - h\Big) = \lim_{h\to 0^+} O(h) = 0.
\end{aligned}$$

Obviously $D^-u(0) = 0$. Since $u(0^+) = 1$ and $u(0^-) = 0$, we therefore obtain that

$$\lim_{\mathbf{Z}\ni m\to\infty} \frac{1}{2}\int_0^{\pi} \frac{2\sin(t/2)}{t} D_m(t)\,dt = \pi \lim_{\mathbf{Z}\ni m\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t) D_m(t)\,dt = \pi\,\frac{1+0}{2} = \frac{\pi}{2}.$$

5.1 ...but it is not absolutely convergent

Note though, that (7) is *not* absolutely convergent. We can see this by rewriting as a series of partial integrals:

$$\int_\pi^\infty \frac{|\sin x|}{|x|}\,dx = \sum_{k=1}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\,dx \ge \sum_{k=1}^{\infty} \frac{1}{k\pi} \int_{k\pi}^{(k+1)\pi} |\sin x|\,dx = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k} = \infty,$$

since $\displaystyle\int_0^\pi |\sin x|\,dx = 2$ (this is the same for every interval $[k\pi,(k+1)\pi]$).

6 An Approximation Result

So this is going to be fairly similar to what we did in lecture 5, but instead of waving our hands, let's go through the details.

Theorem. Suppose that $u \in G(\mathbf{R}) \cap L^2(\mathbf{R})$ and let $\epsilon > 0$. Then there exists a function v in $C^2(\mathbf{R})$ such that the following holds.

(i) There exists an interval $[-M, M]$ such that $v(x) = 0$ for $|x| > M$.

(ii) $\displaystyle\int_{-\infty}^{\infty} |u(x) - v(x)|^2 \, dx < \epsilon^2$.

(iii) $\displaystyle\int_{-\infty}^{\infty} |u(x) - v(x)| \, dx < \epsilon$.

Proof. To produce such a function v, we will use the fact that u and $|u|^2$ are absolutely integrable (in the Riemann sense) to find a partition where any Riemann sum is close enough to the integral. Before doing this, lets fix so we have *compact support*. We do this by observing that since u and $|u|^2$ are absolutely integrable on $\mathbf{R}$, there exists a number $L > 0$ such that

$$\max\left\{\int_{-\infty}^{-L} |u(x)| \, dx + \int_{L}^{\infty} |u(x)| \, dx, \ \int_{-\infty}^{-L} |u(x)|^2 \, dx + \int_{L}^{\infty} |u(x)|^2 \, dx\right\} < \min\left\{\frac{\epsilon}{3}, \ \frac{\epsilon^2}{18}\right\}.$$

Now, on $[-L, L]$, we choose a partition

$$x_0 = -L < x_1 < x_2 < \cdots < x_n = L$$

such that u is continuous on each $]x_k, x_{k+1}[$ and

$$\left|\int_{-L}^{L} u(x) \, dx - \sum_{k=0}^{n-1} c_k (x_{k+1} - x_k)\right| < \frac{\epsilon}{3},$$

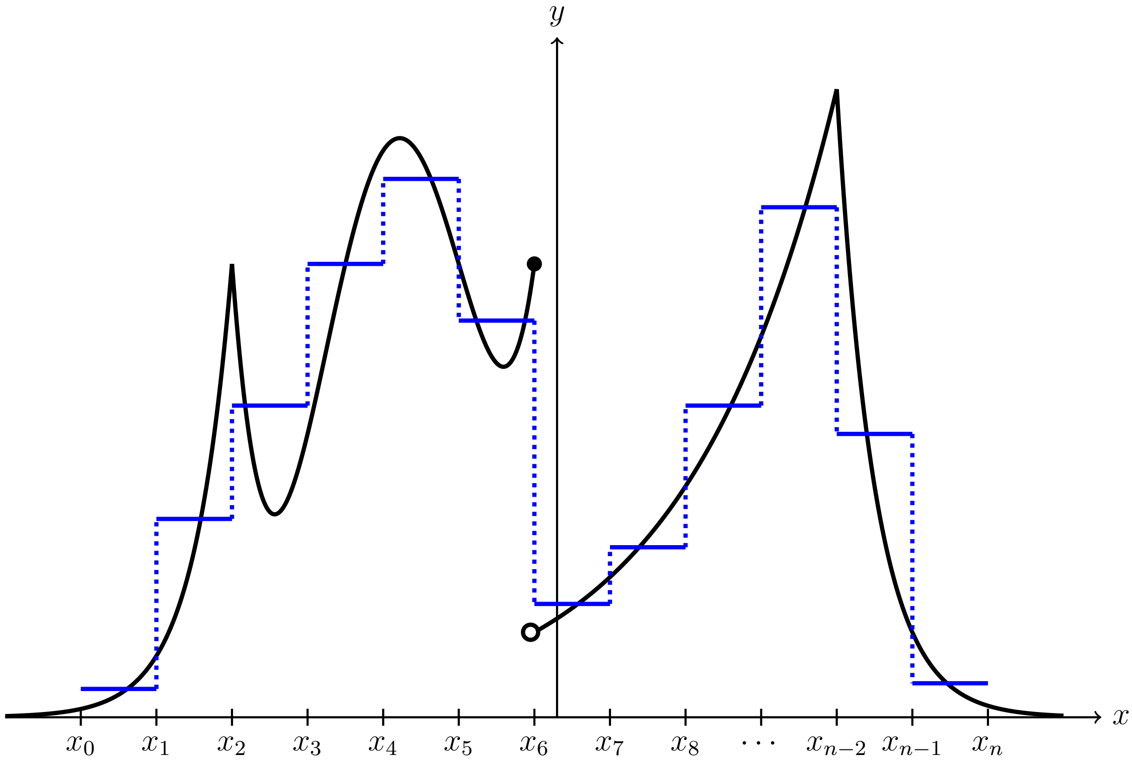
where $c_k = u(\xi_k)$ for some $\xi_k \in [x_k, x_{k+1}]$. Let $\zeta(x) = c_k$ when $x_k \leq x < x_{k+1}$, $k = 0, 1, 2, \ldots$ and zero elsewhere.

Note that by the uniform continuity of u on each $[x_i, x_{i+1}]$ (after possible redefinition at the end points), it is true that for any $\epsilon > 0$, there is a $\delta_i > 0$ such that

$$x, y \in [x_i, x_{i+1}] : |x - y| < \delta_i \quad \Rightarrow \quad |u(x) - u(y)| < \min\left\{\frac{\epsilon}{6L}, \ \frac{\epsilon}{\sqrt{18L}}\right\}.$$

We therefore choose $\delta = \min\{\delta_i\}$ and since clearly $\delta > 0$, it is possible to refine the partition $\{x_i\}_{i=0}^{n}$ of $[-L, L]$ such that $|x_{i+1} - x_i| < \delta$, $i = 0, 1, 2, \ldots, n-1$.

Graphically, we could have something like this.



From this it follows that

$$|u(x) - \zeta(x)| = |u(x) - c_k| \leq \min\left\{\frac{\epsilon}{6L}, \frac{\epsilon}{\sqrt{36L}}\right\}, \quad x_i < x < x_{i+1},$$

since $c_k = u(\xi_k)$ for some ξ_k such that $x_k < \xi_k \leq x_{k+1}$. The inequality might not hold at the end-points, but this does not matter for the integral. This implies that

$$\int_{x_i}^{x_{i+1}} |u(x) - \zeta(x)|\, dx \leq \frac{\epsilon}{6L}\, |x_{i+1} - x_i|, \quad i = 0, 1, 2, \ldots, n-1$$

and

$$\int_{x_i}^{x_{i+1}} |u(x) - \zeta(x)|^2\, dx \leq \frac{\epsilon^2}{36L}\, |x_{i+1} - x_i|, \quad i = 0, 1, 2, \ldots, n-1,$$

so

$$\begin{aligned}
\|u - \zeta\|^2_{L^2(\mathbf{R})} &= \int_{-\infty}^{\infty} |u(x) - \zeta(x)|^2\, dx \\
&= \int_{-\infty}^{-L} |u(x)|^2\, dx + \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} |u(x) - \zeta(x)|^2\, dx + \int_{L}^{\infty} |u(x)|^2\, dx \\
&\leq \frac{\epsilon^2}{18} + \sum_{k=0}^{n-1} \frac{\epsilon^2}{36L} |x_{i+1} - x_i| = \frac{\epsilon^2}{9}
\end{aligned}$$

and

$$\begin{aligned}
\|u - \zeta\|_{L^1(\mathbf{R})} &= \int_{-\infty}^{\infty} |u(x) - \zeta(x)|\, dx \\
&= \int_{-\infty}^{-L} |u(x)|\, dx + \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} |u(x) - \zeta(x)|\, dx + \int_{L}^{\infty} |u(x)|\, dx \\
&\leq \frac{\epsilon}{6} + \sum_{k=0}^{n-1} \frac{\epsilon}{6L}\, |x_{i+1} - x_i| = \frac{\epsilon}{3}
\end{aligned}$$

So how do we turn this into something that's twice differentiable? We will proceed similar to what we did in lecture 5, but a straight line will not do. Suppose we have two constant segments, one defined as 0 on $[-1, 0]$ and one defined as 1 on $[1, 2]$. Can we join these segments smoothly? Sure we can, in a lot of different ways. For our purpose, we need something of class C^2, so twice continuously differentiable. The most straight forward idea is probably to match a polynomial at the end points while making certain that also the derivatives match. Let $\eta(x)$ be such a polynomial. We want the following to hold:

$$\eta(0) = 0, \quad \eta(1) = 1, \quad \eta'(0) = 0, \quad \eta'(1) = 0, \quad \eta''(0) = 0, \quad \eta''(1) = 0.$$

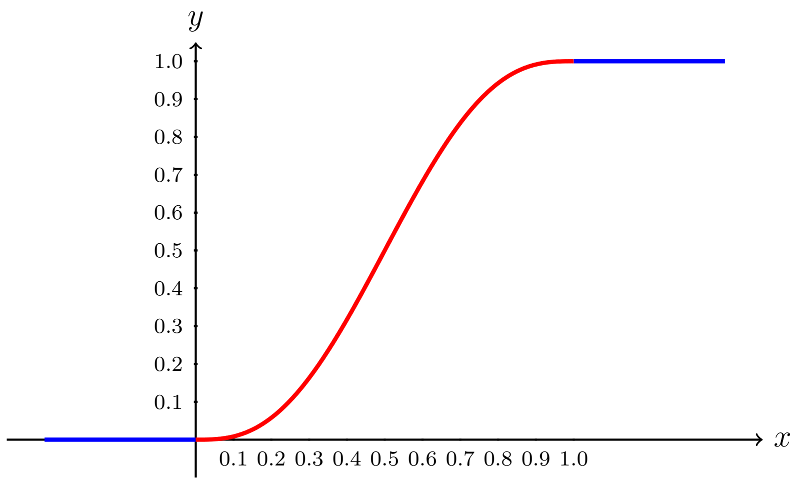
So six restrictions. Using a fifth degree polynomial as ansatz, we find that

$$\eta(x) = x^5 - 15x^4 + 10x^3.$$

Some basic analysis shows that there are no extreme values on $]0, 1[$ so the maximum and minimum are attained at the end points, which is nice since that means that $0 \leq \eta(x) \leq 1$ on $[0, 1]$. Let's make the following definition:

$$\eta(x) = \begin{cases} 0, & x < 0, \\ 6\,x^5 - 15\,x^4 + 10\,x^3, & 0 \leq x \leq 1, \\ 1, & x > 1. \end{cases}$$

What we now have accomplished can be seen in the figure below.



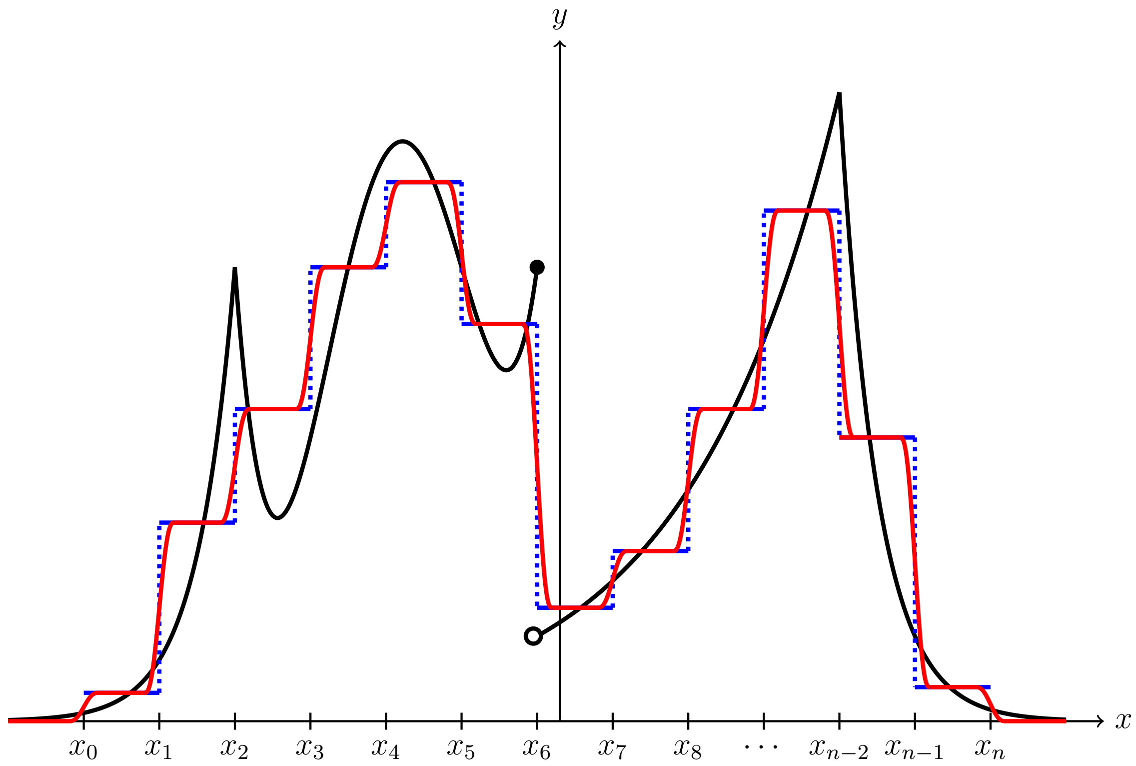
We can use this function in the following way, scaling and translating as needed. Choose a $\delta > 0$ such that (yeah yeah..)

$$\delta < \min\left\{\frac{\epsilon}{12K(n+1)}, \frac{\epsilon^2}{72K^2(n+1)}\right\},$$

where K is some number such that $|\zeta(x)| \leq K$ for all $x \in [-L, L]$. Define v such that $v(x) = c_k$ when $x_k + \delta \leq x \leq x_{k+1} - \delta$ and for $x_k - \delta < x < x_k + \delta$, we use the function

$$c_k + (c_{k+1} - c_k)\eta\left(\frac{x - (x_k - \delta)}{2\delta}\right).$$

The result can be seen in the graph below.



Note that $v(x) = \zeta(x)$ for most of $\mathbf{R}$, so

$$\|v - \zeta\|^2_{L^2(\mathbf{R})} = \int_{-\infty}^{\infty} |v(x) - \zeta(x)|^2 \, dx = \sum_{k=0}^{n} \int_{x_k-\delta}^{x_k+\delta} |v(x) - \zeta(x)|^2 \, dx$$
$$\le 8(n+1)K^2\delta \le \frac{\epsilon^2}{9},$$

where we used the rough estimate $|v(x) - \zeta(x)| \le 2K$ on $[-L, L]$, which holds if $|u(x)| \le K$ (which implies that $|\zeta(x)| \le K$ as well).

Similarly, we obtain that

$$\|v - \zeta\|_{L^1(\mathbf{R})} = \int_{-\infty}^{\infty} |v(x) - \zeta(x)| \, dx = \sum_{k=0}^{n} \int_{x_k-\delta}^{x_k+\delta} |v(x) - \zeta(x)| \, dx$$
$$\le 4(n+1)K\delta \le \frac{\epsilon}{3}.$$