Lecture 8: Uniqueness

Johan Thim (johan.thim@liu.se)

March 23, 2023

"Consider that a divorce!"
—Douglas Quaid

1 Uniqueness

Similar to where we were in Lecture 5 for Fourier series, we now find ourselves in a similar spot with regards to the Fourier transform. Indeed, we have seen conditions for when the Fourier transform exists and we have seen conditions for when we can find the inverse (analogously to when the Fourier series converges "correctly").

Question. Suppose that $u, v \in G(\mathbf{R})$ has the Fourier transforms $\mathcal{F}\,u$ and $\mathcal{F}\,v$, respectively. If $\mathcal{F}\,u = \mathcal{F}\,v$, what can we say about the functions u and v? Are they equal? In what sense?

We will show that if $u, v \in G(\mathbf{R})$ and $\mathcal{F}\,u = \mathcal{F}\,v$, then $u(x) = v(x)$ wherever both u and v are continuous.

2 Cesàro Summation for Integrals

For our purposes, recall that we consider the *principal value* for the Fourier transform and its inverse, that is, integrals of the form

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx, \tag{1}$$

and that this might change for which functions f the integral is convergent. Now, we can obtain even better convergence by considering the mean value integral of the partial integrals, that is,

$$\lim_{M\to\infty} \frac{1}{M} \int_0^M \int_{-r}^{r} f(x)\,dx\,dr. \tag{2}$$

This is analogous to the mean value of the partial sums for the Cesàro summation for series. Similarly to that case, if the limit in (1) exists, then the limit in (2) exists as well and converges to the same value.

Indeed, let $I_r = \displaystyle\int_{-r}^{r} u(x)\,dx \to I$ be convergent and let $\epsilon > 0$. Then there exists $N > 0$ such that $|I_r - I| \leq \epsilon$ if $r \geq N$ and

$$\left|\frac{1}{M}\int_0^M I_r\,dr - I\right| = \left|\frac{1}{M}\int_0^M (I_r - I)\,dr\right| \leq \frac{1}{M}\int_0^N |I_r - I|\,dr + \frac{1}{M}\int_N^M |I_r - I|\,dr.$$

Observing that

$$\int_0^N |I_r - I|\,dr \le \int_0^N \int_{-r}^r |u(x)|\,dx\,dr + NI \le N\int_{-N}^N |u(x)|\,dx + NI < \infty,$$

we find that

$$\lim_{M\to\infty} \frac{1}{M}\int_0^N |I_r - I|\,dr = 0.$$

Since also

$$\frac{1}{M}\int_N^M |I_r - I|\,dr \le \frac{1}{M}\int_N^M \epsilon\,dr \le \frac{M-N}{M}\epsilon < \epsilon,$$

it must be true that

$$\lim_{M\to\infty} \frac{1}{M}\int_0^M I_r\,dr = I.$$

3 The Fejér Kernel for the Fourier Transform

We wish to investigate the limit

$$\lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^R \mathcal{F}\,u(\omega)e^{i\omega x}\,d\omega$$

and see if it exists, and if so, what the limit is (hoping for something similar to $u(x)$). To this end, let's consider the Cesàro means:

$$\begin{aligned}\frac{1}{M}\int_0^M \left(\frac{1}{2\pi}\int_{-r}^r \mathcal{F}\,u(\omega)e^{i\omega x}\,d\omega\right)dr &= \frac{1}{2\pi M}\int_{-M}^M \int_{|\omega|}^M \mathcal{F}\,u(\omega)e^{i\omega x}\,dr\,d\omega \\ &= \frac{1}{2\pi M}\int_{-M}^M \mathcal{F}\,u(\omega)\left(M-|\omega|\right)e^{i\omega x}\,d\omega \\ &= \frac{1}{2\pi}\int_{-M}^M \mathcal{F}\,u(\omega)\left(1-\frac{|\omega|}{M}\right)e^{i\omega x}\,d\omega,\end{aligned}$$

where we changed the order of integration in the first equality. Now, writing out the definition of $\mathcal{F}\,u(\omega)$, we find that

$$\begin{aligned}\frac{1}{2\pi}\int_{-M}^M \mathcal{F}\,u(\omega)\left(1-\frac{|\omega|}{M}\right)e^{i\omega x}\,d\omega &= \frac{1}{2\pi}\int_{-M}^M \left(\int_{-\infty}^\infty u(t)e^{-it\omega}\,dt\right)\left(1-\frac{|\omega|}{M}\right)e^{i\omega x}\,d\omega \\ &= \int_{-\infty}^\infty u(t)\frac{1}{2\pi}\int_{-M}^M \left(1-\frac{|\omega|}{M}\right)e^{i\omega(x-t)}\,d\omega\,dt \\ &= \int_{-\infty}^\infty u(t)F_M(x-t)\,dt = \int_{-\infty}^\infty u(t+x)F_M(t)\,dt,\end{aligned}$$

where we used Fubini's theorem and where

$$F_M(t) = \frac{1}{2\pi}\int_{-M}^M \left(1-\frac{|\omega|}{M}\right)e^{i\omega t}\,d\omega$$

is the Fejér kernel on the real line.

Theorem. For $x \neq 0$, we have

$$F_M(x) = \frac{1 - \cos Mx}{\pi M x^2} = \frac{M}{2\pi}\left(\frac{\sin(Mx/2)}{Mx/2}\right)^2. \tag{3}$$

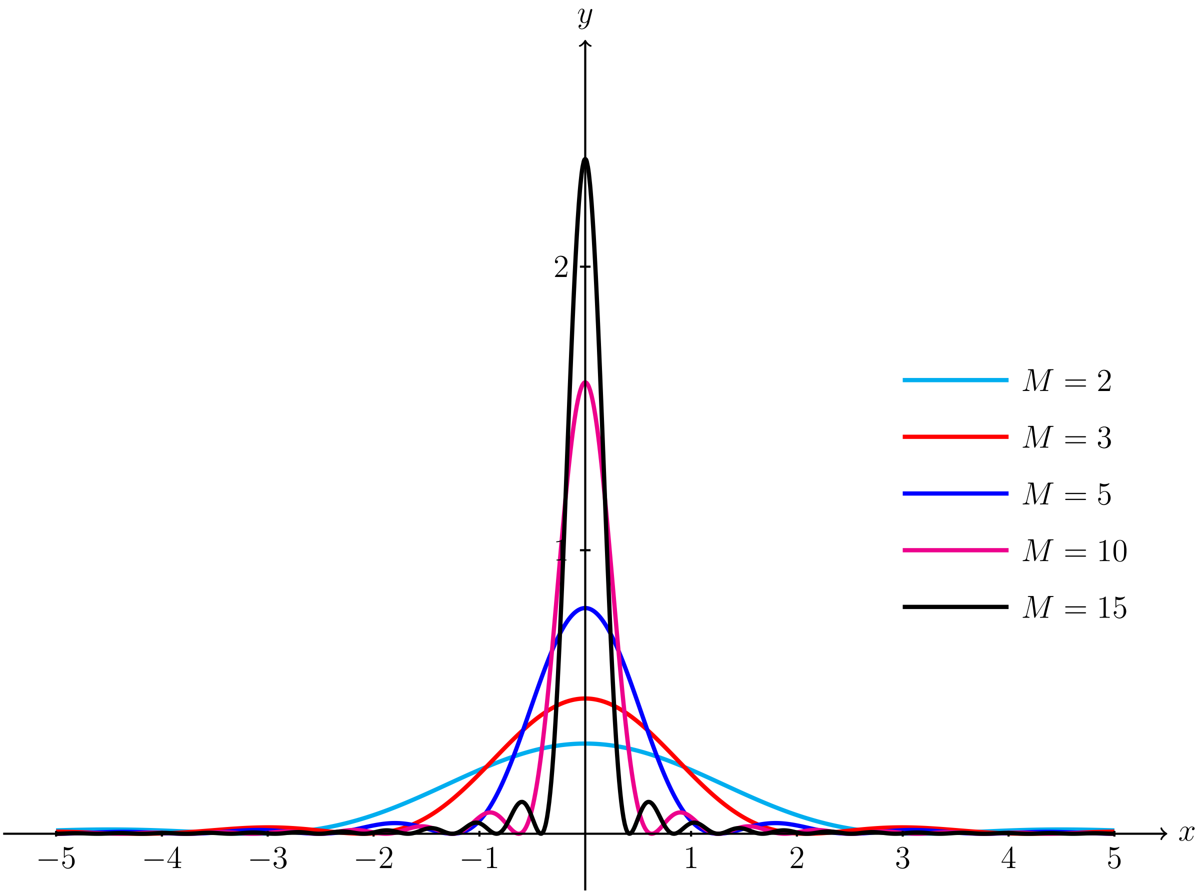
Proof. Using integration by parts, we obtain that for $x \neq 0$,

$$\begin{aligned}\int_{-M}^{M}\left(1 - \frac{|w|}{M}\right) e^{i\omega x}\, d\omega &= \frac{1}{ix}\left(\left[\left(1 - \frac{|w|}{M}\right) e^{i\omega x}\right]_{\omega=-M}^{\omega=M} + \frac{1}{M}\int_{-M}^{M} \operatorname{sgn}(\omega) e^{i\omega x}\, d\omega\right)\\ &= \frac{1}{M(ix)^2}\left(-\left[e^{i\omega x}\right]_{-M}^{0} + \left[e^{i\omega x}\right]_{0}^{M}\right) = \frac{1}{Mx^2}\left(1 - e^{-iM\omega x} - e^{iMx} + 1\right)\\ &= \frac{2 - 2\cos Mx}{Mx^2},\end{aligned}$$

so

$$F_M(x) = \frac{1}{M\pi}\frac{1 - \cos Mx}{x^2}, \quad x \neq 0. \tag{4}$$

Since $2\sin^2 t = 1 - \cos 2t$, $t \in \mathbf{R}$, the second formula in (3) above follows from (4). □



Properties of the Fejér kernel on the real line

Theorem.

(i) $F_M(x) \geq 0$ and F_M is an even function.

(ii) $\displaystyle\int_{-\infty}^{\infty} F_M(x)\,dx = 1.$

(iii) If $\tau > 0$, then $\displaystyle\lim_{M\to\infty} F_M(x) = 0$ uniformly for $|x| \geq \tau$.

(iv) $\displaystyle\int_{|x|\geq\tau} F_M(x)\,dx \to 0$ for any $\tau > 0$.

Proof.

(i) These properties are obvious from the previous theorem.

(ii) To prove this identity, observe that if $\phi(x) = 1 - |x|$ for $|x| < 1$ and $\phi(x) = 0$ for $|x| \geq 1$, then (according to the proof of (3) above with $M = 1$ and ω replaced by $-\omega$)

$$\mathcal{F}\,\phi(\omega) = \int_{-1}^{1} (1 - |x|)e^{-i\omega x}\,dx = \frac{2 - 2\cos\omega}{\omega^2}.$$

Now, since ϕ is continuous (at zero) and $D^{\pm}\phi(0)$ exists, we know that (by Dirichlet's theorem from the previous lecture)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2 - 2\cos\omega}{\omega^2}\,e^{i\omega\cdot 0}\,d\omega = \phi(0) = 1.$$

Hence

$$\begin{aligned}\int_{-\infty}^{\infty} F_M(x)\,dx &= \int_{-\infty}^{\infty} \frac{1}{M\pi}\,\frac{1 - \cos Mx}{x^2}\,dx \\ &= /\ t = Mx\ / = \int_{-\infty}^{\infty} \frac{1}{M\pi}\,\frac{1 - \cos t}{(t/M)^2}\,\frac{dt}{M} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1 - \cos t}{t^2}\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2 - 2\cos t}{t^2}\,dt = 1.\end{aligned}$$

(iii) Observing that (for $x \neq 0$)

$$|F_M(x)| = \frac{M}{2\pi}\left(\frac{\sin(Mx/2)}{Mx/2}\right)^2 \leq \frac{M}{2\pi}\left(\frac{2}{Mx}\right)^2 = \frac{2}{M\pi}\,\frac{1}{x^2}, \tag{5}$$

we see that

$$\sup_{|x|\geq\tau} |F_M(x)| \leq \frac{2}{M\pi}\sup_{|x|\geq\tau}\frac{1}{x^2} = \frac{2}{M\pi\tau^2} \to 0,$$

as $M \to \infty$. Hence we have uniform convergence for $|x| \geq \tau$ for any $\tau > 0$.

(iv) Furthermore, inequality (5) also implies that

$$\int_{\tau}^{\infty} F_M(x)\,dx \leq \frac{2}{M\pi}\int_{\tau}^{\infty}\frac{1}{x^2}\,dx = \frac{2}{M\pi\tau} \to 0,$$

as $M \to \infty$. The integral from $-\infty$ to $-\tau$ is handled analogously. □

Theorem. Suppose that $u \in G(\mathbf{R})$ (so u has right- and lefthand limits at $x \in \mathbf{R}$). Then

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \mathcal{F}\, u(\omega) \left(1 - \frac{|\omega|}{R}\right) e^{i\omega x}\, d\omega = \frac{u(x^+) + u(x^-)}{2}.$$

Proof. Since

$$\frac{1}{2\pi} \int_{-M}^{M} \mathcal{F}\, u(\omega) \left(1 - \frac{|\omega|}{M}\right) e^{i\omega x}\, d\omega = \int_{-\infty}^{\infty} u(t+x) F_M(t)\, dt,$$

proving that

$$\int_0^{\infty} (u(x+t) - u(x^+)) F_M(t)\, dt + \int_{-\infty}^{0} (u(x+t) - u(x^-)) F_M(t)\, dt \to 0,$$

as $M \to \infty$, is sufficient due to the fact that this implies that the Fejér mean converges:

$$\lim_{M\to\infty} \int_{-\infty}^{\infty} u(t) F_M(x-t)\, dt = \frac{u(x^+) + u(x^-)}{2}.$$

Here we use the identities

$$\int_{-\infty}^{0} F_M(t)\, dt = \int_0^{\infty} F_M(t)\, dt = \frac{1}{2}.$$

So, let $\epsilon > 0$. Since u has a right-hand limit at x, there is a $\delta > 0$ such that

$$0 < t < \delta \quad \Rightarrow \quad |u(x+t) - u(x^+)| < \epsilon.$$

We exploit this and the uniform convergence of F_M to obtain that

$$\begin{aligned} \left| \int_0^{\infty} (u(x+t) - u(x^+)) F_M(t)\, dt \right| &\leq \int_0^{\delta} \epsilon F_M(t)\, dt + \int_{\delta}^{\infty} |u(x+t) - u(x^+)| F_M(t)\, dt \\ &\leq \epsilon \int_0^{\infty} F_M(t)\, dt + \int_{\delta}^{\infty} |u(x+t) - u(x^+)| F_M(t)\, dt \\ &\to \frac{\epsilon}{2} \end{aligned}$$

as $M \to \infty$ since F_M converges uniformly to zero on $[\delta, \infty[$, so

$$\begin{aligned} \int_{\delta}^{\infty} |u(x+t) - u(x^+)| F_M(t)\, dt &\leq \left(\sup_{t\geq\delta} F_M(t) \right) \int_{\delta}^{\infty} |u(x+t)|\, dx + |u(x^+)| \int_{\delta}^{\infty} F_M(t)\, dt \\ &\leq \left(\sup_{t\geq\delta} F_M(t) \right) \int_{-\infty}^{\infty} |u(x)|\, dx + |u(x^+)| \int_{\delta}^{\infty} F_M(t)\, dt \to 0, \end{aligned}$$

as $M \to \infty$. The second integral is handled analogously. □

An immediate consequence of this theorem is the following uniqueness result.

Uniqueness

Corollary. Suppose that $u \in G(\mathbf{R})$ and $v \in G(\mathbf{R})$. If $\mathcal{F}\,u(\omega) = \mathcal{F}\,v(\omega)$ for every $\omega \in \mathbf{R}$, then $u(x) = v(x)$ for every $x \in \mathbf{R}$ where both u and v are continuous.

Furthermore, the following corollary is clear since if an integral converges in the usual sense, then the Cesàro-means converge to the same value. This shows that the assumption that the onesided derivatives exist, which we used in the previous lecture, is not necessary. The inversion works anyway for functions in $G(\mathbf{R})$, provided that the limit exists. Of course, knowing that the onesided derivatives exist does ensure that the limit exists, so there is that...

Corollary. Suppose that $u \in G(\mathbf{R})$. Then

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \mathcal{F}\,u(\omega) e^{i\omega x}\, d\omega = \frac{u(x^+) + u(x^-)}{2}, \tag{6}$$

whenever the limit exists.

This means that if the limit exists and u has right- and lefthand limits, then the inversion gives the expected result. It could still be that the limit does not exist, however.

4 The Inverse Fourier Transform?

The last corollary in the previous section is an interesting result. It basically claims that if we try to do Fourier inversion, it works if all limits in the formula exists. Let's make a formal definition of the "operator" $\mathcal{F}^{-1}$.

The Inverse Fourier Transform

Definition. For a reasonable function $U \colon \mathbf{R} \to \mathbf{C}$, we define

$$\mathcal{F}^{-1}\,U(x) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} U(\omega) e^{i\omega x}, \tag{7}$$

for those $x \in \mathbf{R}$ where this makes sense.

We know from the previous section that

$$u \in G(\mathbf{R}) \text{ continuous} \quad \Rightarrow \quad (\mathcal{F}^{-1}\,\mathcal{F}\,u)(x) = u(x),\ x \in \mathbf{R}.$$

If $U \in L^1(\mathbf{R})$, the integral

$$\int_{-\infty}^{\infty} U(\omega) e^{i\omega x}\, d\omega$$

is absolutely convergent so the limit in (7) exists and obviously we can write

$$\mathcal{F}^{-1}\,U(x) = \frac{1}{2\pi} \mathcal{F}(U(\omega))(-x). \tag{8}$$

So the inverse transform is basically given by the Fourier transform. This implies that if $\mathcal{F}u$ is an element in $L^1(\mathbf{R})$, then u actually can't be too discontinuous given that (6) holds and that u basically is a Fourier transform of a function from $L^1(\mathbf{R})$ (in more technical terms this implies that u must be equal to a continuous function *almost everywhere*). We also note that this identity is a better way of proving the identity

$$\mathcal{F}(\mathcal{F}(u))(x) = 2\pi u(-x)$$

for continuous functions u from $G(\mathbf{R})$ such that $\mathcal{F}u \in G(\mathbf{R})$. The equality actually holds at all points where u is continuous even if there are points of discontinuity (it's a local property).

If we consider the principal value when defining the Fourier transform, the equality (8) can be extended to all functions where the limit in (7) exists.

5 The Fourier Transform of a Product

So is there a way of finding the Fourier transform of a product? As it turns out, there is. At least if we are willing to calculate a convolution in the frequency domain (assuming things are defined).

Fourier Transform of a product

Theorem. Suppose that $u, v \in G(\mathbf{R})$ such that $uv, \mathcal{F}u, \mathcal{F}v \in G(\mathbf{R})$. Then

$$\mathcal{F}(uv)(\omega) = \frac{1}{2\pi}\mathcal{F}(u) * \mathcal{F}(v)(\omega). \tag{9}$$

Proof. We observe that since $v \in G(\mathbf{R})$ and $\mathcal{F}v \in G(\mathbf{R})$, it follows from the inversion theorem that we can write, for all $x \in \mathbf{R}$ where v is continuous (which is all except for a countable set),

$$v(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega)e^{i\omega x}\,d\omega$$

if $V(\omega) = \mathcal{F}v(\omega)$. Therefore we obtain that

$$\begin{aligned}
\mathcal{F}(uv(x))(\xi) &= \int_{-\infty}^{\infty} u(x)v(x)e^{-i\xi x}\,dx = \int_{-\infty}^{\infty} u(x)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega)e^{i\omega x}\,d\omega\right) e^{-i\xi x}\,dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega)\int_{-\infty}^{\infty} u(x)e^{-i(\xi-\omega)x}\,dx\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega)U(\xi-\omega)\,d\omega \\
&= \frac{1}{2\pi}(U * V)(\xi),
\end{aligned}$$

where we used Fubini's theorem when changing the order of integration (we know that both u and V are in $G(\mathbf{R})$).

Note that there are a lot of things that need to align correctly for the previous result to hold. Functions and their respective transforms need to belong to $G(\mathbf{R})$, even if that assumption looks unnecessary from the final formula. Let's look at an example where this problem becomes clear.

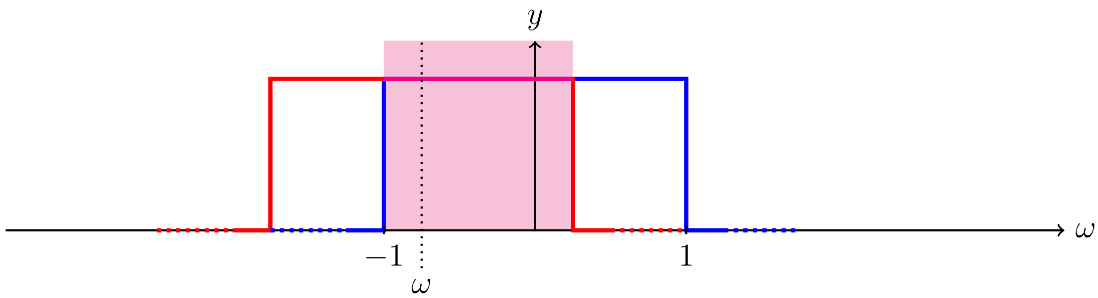
Example

Formally find the Fourier transform of $u(x) = \operatorname{sinc}(x)^2 = \left(\frac{\sin x}{x}\right)^2$.

Solution. Recall that the Fourier transform of $v(x) = 1$ when $-1 \le x \le 1$ and $v(x) = 0$ elsewhere, was $\mathcal{F}\,v(\omega) = 2\operatorname{sinc}(\omega)$. This would indicate that $\mathcal{F}(2\operatorname{sinc}(\omega))(x) = 2\pi v(-x)$, which would mean that $\mathcal{F}(\operatorname{sinc}(x))(\omega) = \pi v(-\omega) = \pi v(\omega)$. Thus $\mathcal{F}(\operatorname{sinc})(\omega) = \pi$ when $|\omega| < 1$ and $\mathcal{F}(\operatorname{sinc})(\omega) = 0$ when $|\omega| \ge 1$. This is a formal result since $\operatorname{sinc}(x)$ does *not* belong to $G(\mathbf{R})$, so our definition of the Fourier transform does not hold. However, the function $\operatorname{sinc}^2(x)$ does belong to $G(\mathbf{R})$, so there exists a Fourier transform of $u(x)$. Proceeding formally, we find that

$$\mathcal{F}(\operatorname{sinc}^2)(\omega) = \frac{1}{2\pi}\,\mathcal{F}(\operatorname{sinc}) * \mathcal{F}(\operatorname{sinc})(\omega) = \begin{cases} \frac{\pi}{2}\big(2 - |\omega|\big), & |\omega| < 2, \\ 0, & |\omega| \ge 2. \end{cases}$$

Why? Well, the procedure is analogous to the example we saw last lecture. We need to calculate the convolution of two identical boxes, so symmetry should almost be enough to assume that it's a triangle but let's do the calculation. Let $F(\omega) = \mathcal{F}(\operatorname{sinc})(\omega)$.



So if $-2 < \omega < 0$, then

$$F * F(\omega) = \int_{-1}^{\omega+1} F(\xi)F(\omega - \xi)\,d\xi = \int_{-1}^{\omega+1} \pi^2\,d\xi = \pi^2(\omega + 2),$$

and if $0 < \omega < 2$, then

$$F * F(\omega) = \int_{\omega-1}^{1} F(\xi)F(\omega - \xi)\,d\xi = \int_{\omega-1}^{1} \pi^2\,d\xi = \pi^2(2 - \omega).$$

For $|w| > 2$ we have $F * F(\omega) = 0$.

So is this really the Fourier transform of $\operatorname{sinc}^2(x)$? One way of proving this is to actually use the inversion formula we derived above, which basically means that we take the Fourier transform of the function $V(\omega) = \pi(2 - |\omega|)/2$ for $|\omega| < 2$ (and zero elsewhere):

$$(\mathcal{F}^{-1}\,V)(x) = \frac{1}{2\pi}\int_{-2}^{2} \frac{\pi}{2}(2 - |\omega|)e^{i\omega x}\,d\omega = \cdots = \frac{1}{4}\,\frac{2 - e^{-i2x} - e^{i2x}}{x^2} = \frac{1}{2}\,\frac{1 - \cos 2x}{x^2} = \operatorname{sinc}^2(x).$$

This operation is allowed since it is clear that $V \in G(\mathbf{R})$ so the Fourier transform is defined as before.

Does this mean that $\mathcal{F}\,u(\omega) = V(\omega)$? It actually does. Observe that $V \in G(\mathbf{R})$, so we can write

$$u(x) = (\mathcal{F}^{-1}\,V)(x) = \frac{1}{2\pi}(\mathcal{F}\,V)(-x).$$

Hence

$$(\mathcal{F}\,u)(\omega) = \frac{1}{2\pi}\,\mathcal{F}\big((\mathcal{F}\,V)(-x)\big)(\omega) = \frac{1}{2\pi}\,\mathcal{F}\big((\mathcal{F}\,V)(x)\big)(-\omega) = \frac{2\pi}{2\pi}V(-(-\omega)) = V(\omega),$$

where we used the fact that $u, V \in G(\mathbf{R})$ and that V is continuous, which implies that taking the Fourier transform twice is equal to 2π times the "mirrored" function (see Section 4).