Lecture 9: The Unilateral Laplace Transform

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"Here's Sub-Zero. Now... Plain Zero!"
—Ben Richards

1 The One Sided Laplace Transform

For reasonable functions, we make the following definition.

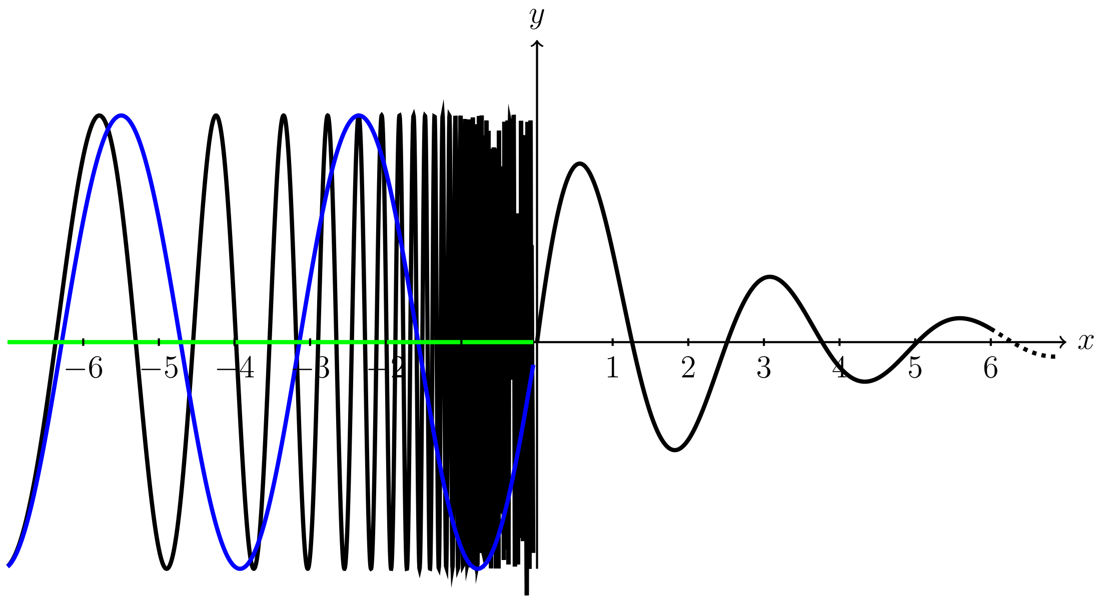
The Laplace transform

Definition. The Laplace transform of $u\colon [0,\infty[\to \mathbf{C}$ is given by

$$\mathcal{L}\, u(s) = \int_0^\infty u(t)e^{-st}\, dt,$$

for those $s \in \mathbf{C}$ where this integral is convergent.

Note that in this definition, we start integrating at $t = 0$. This means that whatever u does for $t < 0$, it is not in any way connected with $\mathcal{L}\, u(s)$. We say that $\mathcal{L}\, u(s)$ is the **one-sided** or **unilateral Laplace transform**.



Black, blue, green... doesn't matter, the Laplace transform will be the same. Therefore we often assume that $u(t) = 0$ for $t < 0$.

Why this restriction? Well, it does make the transform easier to handle. Secondly, there are a lot of applications where we consider the variable t to be *time*, so negative values are not very interesting. Indeed, we assume that something starts at $t = 0$. In other words, we consider *causal* systems. There is a two-sided version of the Laplace transform as well, which is useful in many instances, but in this course we will only use the version above.

Example

Suppose that $u(t) = e^{at}$, where $a \in \mathbf{C}$ is a constant. Show that $\mathcal{L}\, u(s) = \dfrac{1}{s-a}$, $\operatorname{Re} s > \operatorname{Re} a$.

Solution. We find that

$$\mathcal{L}\, u(s) = \int_0^\infty e^{at}e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}, \quad \text{if } \operatorname{Re}(s-a) > 0.$$

As we can see in the example above, the Laplace transform exists if $\operatorname{Re} s > \operatorname{Re} a$. For a function that doesn't grow faster than e^{at} as $t \to \infty$, the Laplace transform will exist at least for $\operatorname{Re} s > \operatorname{Re} a$ (provided that the integral exists). For our purposes, piecewise continuous functions will suffice. Let's make the following definition.

Exponential order (exponential growth)

Definition. We say that the piecewise continuous function $u\colon [0,\infty[$ is of exponential order (of order a) if there exists constants $a > 0$ and $K > 0$ such that $|u(t)| \le Ke^{at}$ for $t \ge 0$. The set of all such functions will be denoted by X_a.

Existence

Theorem. If $u \in X_a$ for some $a > 0$, then the Laplace transform $\mathcal{L}\, u(s)$ exists (at least) for $\operatorname{Re} s > a$. Furthermore,

$$\lim_{L\to\infty} \int_0^L u(t)\, e^{-st}\, dt = \mathcal{L}\, u(s)$$

uniformly and $\mathcal{L}\, u(s)$ is continuous.

Proof. Obviously

$$\left|u(t)e^{-st}\right| \le Ke^{at}\left|e^{-st}\right| = Ke^{at}e^{-t\operatorname{Re} s} = Ke^{-t(\operatorname{Re} s - a)},$$

so if $\operatorname{Re} s > a$ then

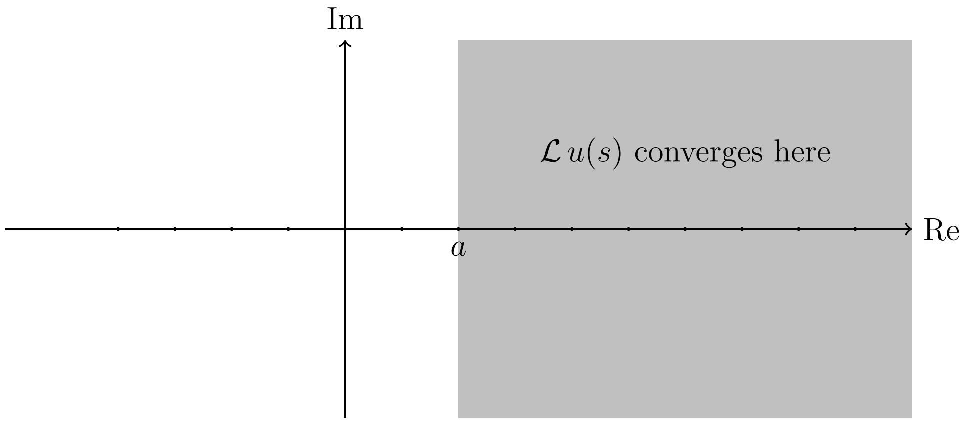
$$\int_0^\infty u(t)e^{-st}\, dt$$

converges absolutely. The fact that the convergence is uniform follows from the inequality above. Indeed, suppose that $\operatorname{Re} s \ge b > a$. Then

$$\begin{aligned}\sup_{\operatorname{Re} s \ge b}\left|\int_0^\infty u(t)\, e^{-st}\, dt - \int_0^L u(t)\, e^{-st}\, dt\right| &\le \sup_{\operatorname{Re} s \ge b}\int_L^\infty \left|u(t)e^{-st}\right|\, dt \\ &\le K \sup_{\operatorname{Re} s \ge b}\int_L^\infty e^{-t(\operatorname{Re} s - a)}\, dt = \frac{Ke^{-L(b-a)}}{b-a} \to 0,\end{aligned}$$

as $L \to \infty$. Since the limit is uniform, we also obtain that $\mathcal{L}\, u(s)$ is continuous. $\square$

So the region of convergence for the one-sided Laplace transform of functions from X_a typically looks like this.



2 Connection to the Fourier Transform?

The Fourier transform is basically a slice of the Laplace transform — where we let $s = i\omega$ — when we restrict the argument function to non-negative values. In other words, we only let s move along the imaginary axis and consider the function $f(t)H(t)$ where $H(t)$ is the Heaviside function. So if u is piecewise continuous and $\mathcal{L}\, u(s)$ exists and $s = \sigma + i\omega$, then

$$\mathcal{L}\, u(s) = \mathcal{F}\big(e^{-\sigma t} u(t) H(t)\big)(\omega).$$

This means that several things we did for the Fourier transform also holds for the Laplace transform, at least when it comes to the calculation of the transforms. The convergence results are different since we now allow exponential growth, but we can "move" the part corresponding to $\operatorname{Re} s$ to the argument function like above and use the corresponding result for the Fourier transform. We'll get back to this.

Qualitatively, one can say that the Fourier transform investigates frequency content in a function by decomposing the function into sinusoids while the Laplace transforms also investigates the amount of exponential growth/decay a function has.

3 Complex Differentiability and Analyticity

Since we're heading into a domain where $s \in \mathbf{C}$, we need to make sure everything is in order. So to this end, let's collect some facts we need. A complex valued function $u \colon \mathbf{C} \to \mathbf{C}$ is called differentiable if

$$u'(z) = \lim_{h \to 0} \frac{u(z+h) - u(z)}{h}$$

exists. The definition is basically the same as for the real case, but going back to the definition of a limit of a complex expression, it is clear that this two dimensional limit is more restrictive than that of the single variable case. Indeed, suppose that $u(z) = \alpha(x, y) + i\beta(x, y)$, where $z = x + iy$. Then the claim that u'_x and u'_y exist is *weaker* than claiming that $u'(z)$ defined as above exists. In fact, if u is differentiable then the components α and β satisfy the Cauchy-Riemann equations:

$$\alpha'_x = \beta'_y \quad \text{and} \quad \alpha'_y = -\beta'_x.$$

Equivalently, these equations can be phrased as

$$i\frac{\partial}{\partial x}u(x+iy) = \frac{\partial}{\partial y}u(x+iy).$$

There are results in the other direction as well. If u is continuous and has partial derivatives that satisfy Cauchy-Riemann's equations, then u is holomorphic (see below).

This topic is not something we need to dig that much deeper into. With the definition above, one can show that this complex derivative satisfies the same "rules" as in the real one-variable case, meaning that the product rule, chain rule, and so on work like expected.

We call a function **holomorphic** at a point z_0 if there is a neighborhood $B(z_0\,;\,\delta)$ of z_0 (meaning some open disc with z_0 as the center) such that f is differentiable for all points in this neighborhood. If one can choose all of $\mathbf{C}$ as the neighborhood, the function is usually referred to as **entire**. Something rather interesting happens here. Remember that the complex derivative requires more to exist than the partial derivatives, so what happens is that if u is holomorphic at z_0 then u is *infinitely differentiable* at z_0. Moreover, it turns out that the function is **analytic** at z_0, meaning that it's complex Taylor series converges to $u(z)$ for z in some neighborhood of z_0. So we have

$$u(z) \text{ holomorphic at } z_0 \quad \Leftrightarrow \quad u(z) = \sum_{k=0}^{\infty} \frac{u^{(k)}(z_0)}{k!}(z-z_0)^k$$

in some neighborhood of z_0. Holomorphic functions have several other very nice properties such as

(i) Cauchy's integral theorem: $\oint_\gamma u(z)\,dz = 0$ for any closed nice enough curve γ;

(ii) Cauchy's integral formula: $u(z) = \dfrac{1}{2\pi i}\oint_\gamma \dfrac{u(\zeta)}{\zeta - z}\,d\zeta$ for $z \in D$ and $\gamma = \partial D$, when u is holomorphic on the closed disc D. This means that the values of u inside D are completely described by the values on the boundary.

(iii) Holomorphic functions are *conformal* if locally invertible (preserving angles).

We could go on and discuss meromorphic functions (holomorphic except for some exception points where we have *poles*) and residue calculus, but that's way off course. We will not use these types of properties, even if some arguments could be made a lot more elegant that way. The power series stuff and complex derivatives are enough.

3.1 The Laplace Transform is Analytic

If $u \in X_a$, the Laplace transform $\mathcal{L}\,u(s)$ is analytic for $\operatorname{Re} s > a$. We can prove this in several ways, and perhaps the most clear one is the following.

Assume first that u is continuous. Let $s = \sigma + i\omega$ and suppose that $a < \sigma_0 < \operatorname{Re} s = \sigma$. Then

$$\mathcal{L}\,u(s) = \int_0^\infty e^{-(\sigma+i\omega)t}u(t)\,dt$$

exists. Put $G(\sigma,\omega,t) = e^{-(\sigma+i\omega)t}u(t)$, $\sigma > \sigma_0$, $t \geq 0$ and $\omega \in \mathbf{R}$. Noting that

$$G'_\sigma(\sigma,\omega,t) = -tG(\sigma,\omega,t) \quad \text{and} \quad G'_\omega(\sigma,\omega,t) = -itG(\sigma,\omega,t),$$

we see that both

$$\int_0^\infty G'_\sigma(\sigma,\omega,t)\,dt = -\int_0^\infty te^{-st}u(t)\,dt \quad \text{and} \quad \int_0^\infty G'_\omega(\sigma,\omega,t)\,dt = -i\int_0^\infty te^{-st}u(t)\,dt$$

converge uniformly since $|te^{-st}| \le Ce^{-\sigma_0 t}$ and the Laplace transform is uniformly convergent for $\operatorname{Re} s > a$. Moreover, if u is continuous then G, G'_σ and G'_ω are all continuous. By Leibniz' rule, this implies that the partial derivatives

$$\frac{\partial}{\partial\sigma}\mathcal{L}\,u(s) = -\mathcal{L}(tu(t))(s) \quad \text{and} \quad \frac{\partial}{\partial\omega}\mathcal{L}\,u(s) = -i\,\mathcal{L}(tu(t))(s)$$

exist and are continuous, which is nice in of itself. However this also shows that the *Cauchy-Riemann equations* hold for $\mathcal{L}\,u(s)$:

$$i\frac{\partial}{\partial\sigma}\mathcal{L}\,u(s) = \frac{\partial}{\partial\omega}\mathcal{L}\,u(s),$$

which proves that $\mathcal{L}\,u(s)$ is analytic at the point s, so $\mathcal{L}\,u(s)$ is analytic for $\operatorname{Re} s > a$ (since the partial derivatives as well as $\mathcal{L}\,u(s)$ were continuous). If u is only piecewise continuous, one can proceed as in the proof of $i\,\mathcal{F}(xu(x)) = U'(\omega)$ from lecture 6.

So while we won't use the analyticity of the Laplace transform directly in this course, the following result will prove useful.

Time multiplication

Theorem. Let $u \in X_a$. Then $\mathcal{L}(tu(t))(s) = -\frac{d}{ds}\mathcal{L}(u(t))(s)$, $\operatorname{Re} s > a$.

Proof. Similarly to the case with the Fourier transform, observe that

$$\frac{d}{ds}\mathcal{L}\,u(s) = \frac{d}{ds}\int_0^\infty u(t)e^{-st}\,dt = /\ \text{Lebiniz's rule}\ / = \int_0^\infty u(t)\frac{d}{ds}e^{-st}\,dt$$
$$= \int_0^\infty -tu(t)e^{-st}\,dt = -\mathcal{L}(tu(t))(s),$$

where Leibniz's rule is applicable due to the argument above.

4 Rules for the Laplace Transform

The fact that the Laplace transform is an integral immediately proves that it is a linear operator.

Linearity

Theorem. If $a, b \in \mathbf{C}$ are constants, then $\mathcal{L}(au(t) + bv(t)) = a\,\mathcal{L}\,u + b\,\mathcal{L}\,v$, whenever $\mathcal{L}\,u$ and $\mathcal{L}\,v$ exists.

Example

Find the Laplace transforms of $\sin t$, $\cos t$ and $t\cos t$.

Solution. By Euler's equations, we obtain that

$$\mathcal{L}(\sin t)(s) = \mathcal{L}\left(\frac{e^{it} - e^{-it}}{2i}\right) = \frac{1}{2i}\left(\mathcal{L}(e^{it}) - \mathcal{L}(e^{-it})\right) = \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) = \frac{1}{2i}\,\frac{2i}{s^2+1}$$
$$= \frac{1}{s^2+1}, \quad \operatorname{Re} s > 0.$$

Similarly, it follows that

$$\mathcal{L}(\cos t)(s) = \frac{s}{s^2+1}, \quad \operatorname{Re} s > 0.$$

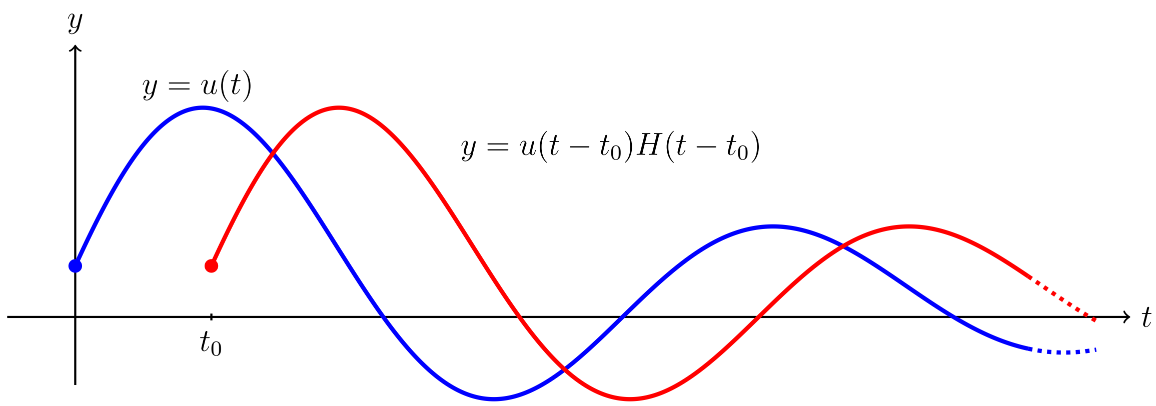
To find the Laplace transform of $t\cos t$, we note that

$$\mathcal{L}(t\cos t)(s) = -\frac{d}{ds}\,\mathcal{L}(\cos t)(s) = -\frac{d}{ds}\frac{s}{s^2+1} = -\frac{s^2+1-2s^2}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}, \quad \operatorname{Re} s > 0.$$

Translation (time shift)

Theorem. If $t_0 > 0$ and $U(s) = \mathcal{L}\,u(s)$, then $\mathcal{L}(u(t-t_0)H(t-t_0))(s) = e^{-st_0}U(s)$.

The factor $H(t-t_0)$ is important since we are working with the unilateral Laplace transform.



Notice the difference with the expression $u(t)H(t-t_0)$ (how would this look?).

Proof. A simple substitution shows that

$$\mathcal{L}(u(t-t_0)H(t-t_0))(s) = \int_{t_0}^{\infty} u(t-t_0)e^{-st}\,dt = /\; y = t - t_0 \;/ = \int_0^{\infty} u(y)e^{-s(y+t_0)}\,dy$$
$$= e^{-st_0}\int_0^{\infty} u(y)e^{-sy}\,dy = e^{-st_0}\,\mathcal{L}(u(t))(s). \qquad \square$$

Let σ_u be the smallest number such that $\mathcal{L}\,u(s)$ converges for $\sigma > \sigma_u$ (which exists if $u \in X_a$).

Scaling

Theorem. If $a > 0$, then $\mathcal{L}(u(ax))(\omega) = \dfrac{1}{a}\,\mathcal{L}(u(t))\left(\dfrac{s}{a}\right)$, $\operatorname{Re} s > a\sigma_u$.

Notice that we only do this for $a > 0$ (why?).

Proof. We see that

$$\mathcal{L}(u(at))(s) = \int_0^\infty u(at)e^{-st}\,dt = /\ y = at\ / = \int_0^\infty u(y)e^{-sy/a}\,\frac{dy}{a}$$
$$= \frac{1}{a}\int_0^\infty u(y)e^{-(s/a)y}\,dy = \frac{1}{a}\,\mathcal{L}\,u\left(\frac{s}{a}\right).$$

□

s-shift

Theorem. If $a \in \mathbf{C}$, then $\mathcal{L}(e^{at}u(t))(s) = (\mathcal{L}(u(t)))(s-a)$, $\operatorname{Re} s > \operatorname{Re} a + \sigma_u$

Proof. We note that

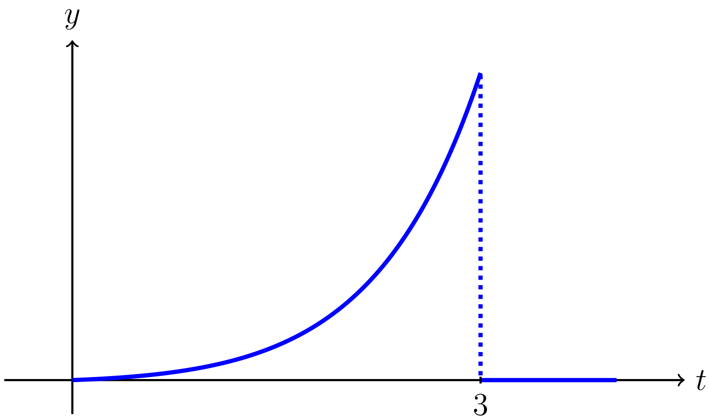
$$\mathcal{L}(e^{at}u(t))(s) = \int_0^\infty u(t)e^{at}e^{-st}\,dt = \int_0^\infty u(t)e^{-(s-a)t}\,dt = (\mathcal{L}\,u)(s-a),$$

which completes the proof. □

Example

Find the Laplace transform of $u(t) = t\,e^{2t}\,H(3-t)$.

Solution. This is some type of exponential function that's cut off at $t = 3$.



We could plug this into the formula and just do the integration, but we can also apply the rules we now know. So, let's call $v(t) = t\,H(3-t)$ so $u(t) = e^{2t}v(t)$. We first observe that

$$v(t) = tH(3-t) = t\big(H(t) - H(t-3)\big) = t\big(H(t) - H(t-3)H(t-3)\big).$$

Why $H(t-3)H(t-3)$? Because obviously $H(t-3)H(t-3) = H(t-3)$ and we want to use the formula $\mathcal{L}(w(t-3)H(t-3)) = e^{-3s}\,\mathcal{L}\,w(s)$. Hence

$$\mathcal{L}(tH(3-t))(s) = -\frac{d}{ds}\left(\mathcal{L}(H(t))(s) - e^{-3s}\,\mathcal{L}(H(t))(s)\right) = -\frac{d}{ds}\left(\frac{1}{s}\left(1 - e^{-3s}\right)\right)$$
$$= \frac{1}{s^2}\left(1 - e^{-3s}\right) - \frac{3e^{-3s}}{s}.$$

Then

$$\mathcal{L}\,u(s) = \mathcal{L}\,v(s-2) = \frac{1 - e^{-3(s-2)}}{(s-2)^2} - \frac{3e^{-3(s-2)}}{s-2} = \frac{1 + e^{6-3s}(5-3s)}{(s-2)^2}.$$

4.1 Differentiation

One of the major uses for the Laplace transform is how it handles derivatives.

Differentiation

Theorem. Let $u \in X_a$ be continuous such that u' is piecewise continuous. Then

$$\mathcal{L}(u'(t))(s) = s\,\mathcal{L}(u(t))(s) - u(0), \quad \operatorname{Re} s > a.$$

Proof. Let $L > 0$ and let

$$x_0 = 0 < x_1 < x_2 < x_3 < \cdots < x_n = L$$

be the points of discontinuity for u' on $[0, L]$. For $k = 0, 1, 2, \ldots, n-1$, we have

$$\begin{aligned}\int_{x_k}^{x_{k+1}} u'(t)e^{-st}\,dt &= \big[u(t)e^{-st}\big]_{t=x_k}^{t=x_{k+1}} + s\int_{x_k}^{x_{k+1}} u(t)e^{-st}\,dt \\ &= u(x_{k+1})e^{-sx_{k+1}} - u(x_k)e^{-sx_k} + s\int_{x_k}^{x_{k+1}} u(t)e^{-st}\,dt,\end{aligned}$$

so

$$\begin{aligned}\int_0^L u'(t)e^{-st}\,dt &= \sum_{k=0}^{n-1}\big(u(x_{k+1})e^{-sx_{k+1}} - u(x_k)e^{-sx_k}\big) + s\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} u(t)e^{-st}\,dt \\ &= \big/ \text{ telescoping sum } \big/ = u(L)e^{-sL} - u(0) + s\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} u(t)e^{-st}\,dt \\ &= u(L)e^{-sL} - u(0) + s\int_0^L u(t)e^{-st}\,dt \to s\,\mathcal{L}\,u(s) - u(0),\end{aligned}$$

since $u(L)e^{-sL} \to 0$ as $L \to \infty$. The last part is clear due to the fact that

$$\left|u(L)e^{-sL}\right| \le Ke^{aL}e^{-L\operatorname{Re} s} \to 0,$$

as $L \to \infty$ ($\operatorname{Re} s > a$). □

Example

Use the Laplace transform to find a solution to $y'(t) + 3y(t) = 0$, $t > 0$, such that $y(0) = 2$.

Solution. Taking the Laplace transform, we find that

$$sY(s) - y(0) + 3Y(s) = 0 \quad \Leftrightarrow \quad Y(s)(s+3) = 2 \quad \Leftrightarrow \quad Y(s) = \frac{2}{s+3},$$

if $\operatorname{Re} s > -3$. We know that $\mathcal{L}(e^{-3t}) = 1/(s+3)$, so $y(t) = 2e^{-3t}$ is a possible solution. Direct verification shows that this solves the equation in question.

If the function $u \in X_a$ has higher order derivatives (that belong to X_a), one can repeatedly apply the previous theorem to transform higher order derivatives. Indeed, if $u^{(n)}$ is piecewise continuous, then this will hold for all derivatives $u^{(k)}$ as well for $k = 0, 1, 2, \ldots, n$. Thus we obtain the following result.

Higher order derivatives

Corollary. Let $u \in X_a$ be a continuous function such that $u^{(n)}$ is piecewise continuous and $u', u'', \dots, u^{(n-1)} \in X_a$. Then

$$\begin{aligned}\mathcal{L}(u^{(n)})(s) &= s^n \mathcal{L}\, u(s) - \sum_{k=0}^{n-1} s^{n-1-k} u^{(k)}(0) \\ &= s^n \mathcal{L}\, u(s) - s^{n-1}u(0) - s^{n-2}u'(0) - \cdots - su^{(n-2)}(0) - u^{(n-1)}(0), \quad \operatorname{Re} s > a.\end{aligned}$$

Example

Find a solution to $y''(t) - 4y(t) = 4e^{2t}$, $t > 0$, with $y(0) = 1$ and $y'(0) = 0$.

Solution. Taking the Laplace transform of both sides in the equality, we find that

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{4}{s-2} \quad &\Leftrightarrow \quad Y(s)(s^2-4) = s + \frac{4}{s-2} \\ &\Leftrightarrow \quad Y(s) = \frac{s}{s^2-4} + \frac{4}{(s^2-4)(s-2)}, \quad \operatorname{Re} s > 2.\end{aligned}$$

Using partial fractions, we find that this expression is equal to

$$\frac{1}{2}\left(\frac{1}{s-2} + \frac{1}{s+2}\right) + \frac{1/4}{s-2} + \frac{1}{(s-2)^2} - \frac{1/4}{s+2}.$$

We see that

$$\mathcal{L}(e^{2t}) = \frac{1}{s-2}, \quad \operatorname{Re} s > 2, \qquad \text{and} \qquad \mathcal{L}(e^{-2t}) = \frac{1}{s+2}, \quad \operatorname{Re} s > -2.$$

Next we note that

$$\frac{1}{(s-2)^2} = -\frac{d}{ds}\left(\frac{1}{s-2}\right)$$

and by the time multiplication theorem,

$$\mathcal{L}(te^{2t}) = -\frac{d}{ds}\left(\frac{1}{s-2}\right), \quad \operatorname{Re} s > 2.$$

Hence

$$\mathcal{L}\left(\frac{1}{4}e^{2t} + te^{2t} + \frac{3}{4}e^{-2t}\right) = Y(s), \quad \operatorname{Re} s > 2,$$

so

$$y(t) = \frac{1}{4}e^{2t} + te^{2t} + \frac{3}{4}e^{-2t}$$

for $t > 0$. Is this the solution? Can we know this without directly verifying?

Hyperbolic functions

Note that $\mathcal{L}(\cosh(t)) = \dfrac{s}{s^2-1}$ and $\mathcal{L}(\sinh(t)) = \dfrac{1}{s^2-1}$ if $\operatorname{Re} s > 1$.