Lecture 10: Convolution and Inversion

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"Don't disturb my Friend. He's dead tired."
—John Matrix

1 Convolution

In the case when we have two functions $u, v\colon [0,\infty[\to \mathbf{C}$, the convolution gets slightly easier to handle.

Convolution

Definition. The convolution $u * v\colon [0,\infty[\to \mathbf{C}$ of $u\colon [0,\infty[\to \mathbf{C}$ and $v\colon [0,\infty[\to \mathbf{C}$ is defined by

$$(u*v)(t) = \int_0^t u(y)v(t-y)\,dy, \quad 0 \le t < \infty,$$

whenever this integral exists.

Theorem. If $u, v\colon [0,\infty[\to \mathbf{C}$ belong to X_a, then $u * v \in X_b$ for every $b > a$ and

$$|u*v(t)| \le Kte^{at}, \quad t > 0.$$

Furthermore,

$$\mathcal{L}(u*v) = \mathcal{L}(u)\,\mathcal{L}(v), \quad \operatorname{Re} s > a.$$

Proof. By definition, $|u(t)| \le K_1e^{at}$ and $|v(t)| \le K_2e^{at}$, so

$$\begin{aligned}|u*v(t)| &= \left|\int_0^t u(\tau)v(t-\tau)\,d\tau\right| \le \text{/ monotonicity /} \le \int_0^t |u(\tau)||v(t-\tau)|\,d\tau \\ &\le \int_0^t Ke^{a\tau}e^{a(t-\tau)}\,d\tau = K_1K_2te^{at},\end{aligned}$$

and since $\lim_{t\to\infty} te^{-\delta t} = 0$ for any $\delta > 0$, it follows that $|u*v(t)| \le Ke^{bt}$ for every $b > a$.

So the convolution of u and v is defined and belongs to X_b. Taking the Laplace transform, we observe that

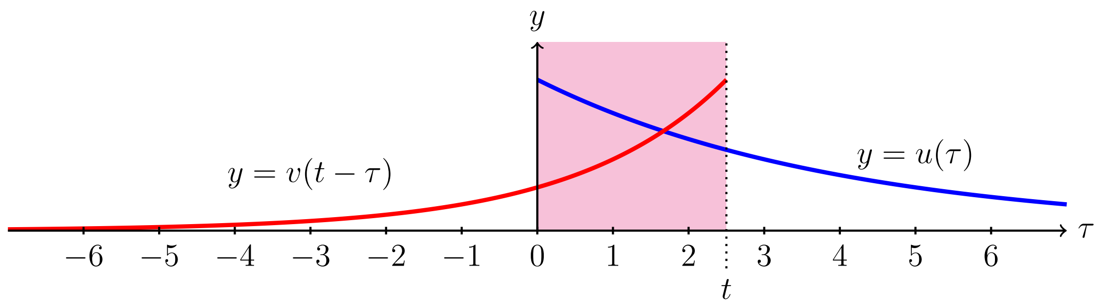
$$\begin{aligned}\mathcal{L}(u*v)(s) &= \int_0^\infty e^{-st}\int_0^t u(\tau)v(t-\tau)\,d\tau\,dt = \int_0^\infty e^{-s(\tau+(t-\tau))}\int_0^t u(\tau)v(t-\tau)\,d\tau\,dt \\ &= /\text{ Fubini }/ = \int_0^\infty u(\tau)e^{-s\tau}\int_\tau^\infty v(t-\tau)e^{-s(t-\tau)}\,dt\,d\tau \\ &= /\ t=y+\tau\ / = \int_0^\infty u(\tau)e^{-s\tau}\int_0^\infty v(y)e^{-sy}\,dy\,d\tau \\ &= \mathcal{L}\,v(s)\int_0^\infty u(\tau)e^{-s\tau}d\tau = \mathcal{L}\,v(s)\,\mathcal{L}\,u(s).\end{aligned}$$

□

Example

Let $u(t) = e^{-t}$ for $t \geq 0$ and $v(t) = e^{-2t}$ for $t \geq 0$. Find $u * v$ and $\mathcal{L}(u*v)(s)$.

Solution. Method 1: direct calculation. First, let's draw the graphs and then mirror the one for v.



Since both u and v are assumed to be zero for negative arguments, the situation is a bit easier than the general convolution we saw when dealing with the Fourier transform. It's only for arguments between zero and t that we obtain something non-zero. Therefore,

$$\begin{aligned}\int_0^t u(\tau)v(t-\tau)\,d\tau &= \int_0^t e^{-\tau}e^{-2(t-\tau)}\,d\tau = \int_0^t e^{-2t+\tau}\,d\tau = e^{-2t}\int_0^t e^{\tau}\,d\tau \\ &= e^{-2t}\left[e^{\tau}\right]_0^t = e^{-2t}(e^t-1) = e^{-t}-e^{-2t},\end{aligned}$$

so

$$\begin{aligned}\mathcal{L}(u*v)(s) &= \mathcal{L}(e^{-t}-e^{-2t}) = \mathcal{L}(e^{-t}) - \mathcal{L}(e^{-2t}) = \frac{1}{s+1}-\frac{1}{s+2} = \frac{s+2-(s+1)}{(s+1)(s+2)} \\ &= \frac{1}{(s+1)(s+2)}, \quad \operatorname{Re} s > -1.\end{aligned}$$

Method 2: Use the convolution theorem. We find that $\mathcal{L}\,u(s) = \dfrac{1}{s+1}$ and $\mathcal{L}\,v(s) = \dfrac{1}{s+2}$, so

$$\mathcal{L}(u*v)(s) = \mathcal{L}\,u(s)\,\mathcal{L}\,v(s) = \frac{1}{(s+1)(s+2)}, \quad \operatorname{Re} s > -1.$$

2 Periodic Functions

Suppose that there exists some $T > 0$ such that $u(t+T) = u(t)$ for every $t \geq 0$. Assuming that u is piecewise continuous, taking the Laplace transform of u yields

$$\begin{aligned}
\mathcal{L}\, u(s) &= \int_0^\infty u(t)e^{-st}\,dt = \sum_{k=0}^\infty \int_{kT}^{(k+1)T} u(t)e^{-st}\,dt \\
&= /\ \tau = t - kT\ / = \sum_{k=0}^\infty \int_0^T u(\tau + kT)e^{-s(\tau+kT)}\,d\tau \\
&= \sum_{k=0}^\infty e^{-skT} \int_0^T u(\tau)e^{-s\tau}\,d\tau = \left(\sum_{k=0}^\infty e^{-skT}\right) \int_0^T u(\tau)e^{-s\tau}\,d\tau \\
&= \frac{1}{1-e^{-sT}} \int_0^T u(\tau)e^{-s\tau}\,d\tau, \quad \operatorname{Re} s > 0,
\end{aligned}$$

where we used the fact that u is periodic and calculated the resulting geometric series.

Example

Let $u(t) = t$, $0 \leq t < 1$, and $u(t+1) = u(t)$ for $t \geq 0$. Find $\mathcal{L}\, u(s)$.

Solution. Since u is periodic with $T = 1$, we find that

$$\begin{aligned}
\mathcal{L}\, u(s) &= \frac{1}{1-e^{-s}} \int_0^T \tau e^{-s\tau}\,d\tau = \frac{1}{1-e^{-s}} \left(\left[\frac{\tau e^{-s\tau}}{-s}\right]_0^1 + \frac{1}{s}\int_0^T e^{-s\tau}\,d\tau \right) \\
&= \frac{1}{1-e^{-s}} \left(\frac{e^{-s}}{-s} + \frac{1}{s}\left[\frac{e^{-s\tau}}{-s}\right]_0^1 \right) = \frac{1}{1-e^{-s}} \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} \right) \\
&= \frac{e^s}{e^s - 1} \left(-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} \right) = \frac{e^s - (s+1)}{s^2(e^s - 1)}.
\end{aligned}$$

Example

Using the periodicity, find the Laplace transform of $u(t) = e^{it}$.

Solution. Since u has period 2π, we find that

$$\begin{aligned}
\mathcal{L}\, u(s) &= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{i\tau}e^{-s\tau}\,d\tau = \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{(i-s)\tau}}{i-s}\right]_0^{2\pi} = \frac{1}{1-e^{-2\pi s}} \cdot \frac{e^{2\pi(i-s)} - 1}{i-s} \\
&= \frac{1}{1-e^{-2\pi s}} \cdot \frac{e^{-2\pi s} - 1}{i-s} = \frac{1}{s-i}, \quad \operatorname{Re} s > 0,
\end{aligned}$$

which is precisely what the transform of e^{at} was derived to be (with $a = i$).

3 Inversion of the Laplace Transform

Similar to the case with the Fourier transform, there's a formula for the inversion of the Laplace transform. We will not use this integral explicitly, but rather use tables to find the correct inverse for a given expression. However, the fact that we have an inversion result means that we know certain uniqueness properties of the Laplace transform. This is an important fact also when using tables.

Laplace inversion formula

Theorem. If $u \in X_a$ has right- and lefthand limits at a point $t > 0$, then

$$\lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \mathcal{L}\, u(\sigma + i\omega) e^{\sigma t} e^{i\omega t}\, d\omega = \frac{u(t^+) + u(t^-)}{2}$$

whenever the limit exists, where the vertical line $\operatorname{Re} z = \sigma$ is contained in the region of convergence of $\mathcal{L}\, u(s)$ ($\sigma > a$ is sufficient).

There are several conditions for the limit to exist. One such example is that $D^{\pm}u(t)$ exists.

Proof. Since $e^{-\sigma t}$ is continuous, it is clear that $v(t) = H(t)u(t)e^{-\sigma t}$ has left- and righthand limits at all $t > 0$ and belongs to $G(\mathbf{R})$. The fact that $\mathcal{L}\, u(\sigma + i\omega) = \mathcal{F}(e^{-\sigma t}u(t)H(t))(\omega)$ enables us to use Fourier inversion, obtaining that

$$\begin{aligned}
\lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \mathcal{L}\, u(\sigma + i\omega) e^{(\sigma+i\omega)t}\, d\omega &= e^{\sigma t} \lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} (\mathcal{F}\, v)(\omega) e^{i\omega t}\, d\omega \\
&= e^{\sigma t} \frac{v(t^+) + v(t^-)}{2} = e^{\sigma t} \frac{e^{-\sigma t}u(t^+) + e^{-\sigma t}u(t^-)}{2} \\
&= \frac{u(t^+) + u(t^-)}{2},
\end{aligned}$$

which is precisely what was stated in the theorem. □

Note: at $t = 0$, the inversion formula yields $u(0^+)/2$.

Similarly with the Fourier transform, a consequence of this result is the following uniqueness theorem.

Uniqueness

Corollary. Suppose that $\mathcal{L}\, u(s)$ and $\mathcal{L}\, v(s)$ are convergent for $\operatorname{Re} s > a$, where $a > 0$ (for example if $u, v \in X_a$). If $\mathcal{L}\, u(s) = \mathcal{L}\, v(s)$ on some vertical line $\operatorname{Re} s = \sigma$, then $u(t) = v(t)$ for all t where u and v are continuous.

Note that we could employ the Fourier result from Lecture 6 as well if $D^{\pm}u(t)$ exists, yielding similar results.

So what use is this in practice? Well, a lot actually. Even if it mostly happens implicitly. Consider the differential equations we've been working with. We solve the equation in the Laplace domain, then find something that gives this Laplace transform and boom. We're done. Or? Well, if we have a uniqueness result, that would be the case (provided that your solution satisfies the required properties). Let's take a look at an example.

Example

Solve the equation

$$y'(t) + y(t) = \begin{cases} \cos t, & 0 \le t \le \frac{\pi}{2}, \\ 1 - \sin t, & t > \frac{\pi}{2}, \end{cases}$$

if $y(0) = 7$.

Solution. Assume that $y \in X_a$. This is important. We will only find solutions that are bounded by some exponential function. Next we reformulate the right-hand side as

$$\begin{aligned}
\left(1 - H\left(t - \frac{\pi}{2}\right)\right)\cos t + H\left(t - \frac{\pi}{2}\right)(1 - \sin t) &= \cos t + H\left(t - \frac{\pi}{2}\right)(1 - \sin t - \cos t) \\
&= \cos t + H\left(t - \frac{\pi}{2}\right)\left(1 - \sin\left(t - \frac{\pi}{2} + \frac{\pi}{2}\right) - \cos\left(t - \frac{\pi}{2} + \frac{\pi}{2}\right)\right) \\
&= \cos t + H\left(t - \frac{\pi}{2}\right)\left(1 - \cos\left(t - \frac{\pi}{2}\right) + \sin\left(t - \frac{\pi}{2}\right)\right).
\end{aligned}$$

The reason for the reformulation is to use the fact that $\mathcal{L}(u(t - t_0)H(t - t_0)) = e^{-st_0}\,\mathcal{L}\,u(s)$. Alternatively, one could perform the integration yielding the transform. Hence the equation has the Laplace transform

$$sY(s) - 7 + Y(s) = \frac{s}{s^2 + 1} + e^{-\pi s/2}\left(\frac{1}{s} - \frac{s}{s^2 + 1} + \frac{1}{s^2 + 1}\right),$$

so if $\operatorname{Re} s > 0$,

$$\begin{aligned}
Y(s) &= \frac{7}{s + 1} + \frac{s}{(s + 1)(s^2 + 1)} + e^{-\pi s/2}\left(\frac{1}{s(s + 1)} + \frac{1 - s}{(s + 1)(s^2 + 1)}\right) \\
&= \frac{7}{s + 1} + \frac{1}{2}\left(\frac{s}{s^2 + 1} + \frac{1}{s^2 + 1} - \frac{1}{s + 1}\right) + e^{-s\pi/2}\left(\frac{1}{s} - \frac{1}{s + 1} + \frac{1}{s + 1} - \frac{s}{s^2 + 1}\right).
\end{aligned}$$

We now observe that

$$\begin{aligned}
Y(s) &= \frac{13}{2}\mathcal{L}(e^{-t}) + \frac{1}{2}(\mathcal{L}(\cos t) + \mathcal{L}(\sin t)) + e^{-s\pi/2}(\mathcal{L}(H(t)) - \mathcal{L}(\cos t)) \\
&= \mathcal{L}\left(\frac{13}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t + H\left(t - \frac{\pi}{2}\right)\left(1 - \cos\left(t - \frac{\pi}{2}\right)\right)\right)
\end{aligned}$$

so by uniqueness we claim that

$$\begin{aligned}
y(t) &= \frac{13}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t + H\left(t - \frac{\pi}{2}\right)\left(1 - \cos\left(t - \frac{\pi}{2}\right)\right) \\
&= \frac{13}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t + H\left(t - \frac{\pi}{2}\right)(1 - \sin t) \\
&= \begin{cases} \frac{13}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t, & 0 < t < \frac{\pi}{2}, \\ \frac{13}{2}e^{-t} + 1 + \frac{1}{2}\cos t - \frac{1}{2}\sin t, & t \ge \frac{\pi}{2}, \end{cases}
\end{aligned}$$

which is OK since $y \in X_a$ with $a > 0$.

Consider $y' - 2ty = 0$, $y(0) = 1$. Using an integrating factor, we know that $y(t) = e^{t^2}$. *Wrongly* assuming that the solution belongs to X_a for some $a > 0$, we would find that (ignoring any issues with the complex variable),

$$sY(s) - 1 + 2Y'(s) = 0 \quad \Leftrightarrow \quad Y(s) = Ce^{-s^2/4} + se^{-s^2/4}.$$

This is *NOT* the Laplace transform of e^{t^2} (for any constant C). Nope. Nein. Niet. Be very careful when using the uniqueness result!

4 Limit Results

Final value theorem

Theorem. Suppose that $u\colon [0,\infty[\to \mathbf{C}$ is a bounded function and that $u(t) \to A$ as $t \to \infty$. Then $A = \lim_{\mathbf{R} \ni s \to 0^+} s\,\mathcal{L}\,u(s)$.

Proof. Let $s \geq 0$ (so s is real). We use the fact that u is bounded to obtain uniform convergence. The Laplace transform of u exists and

$$s\,\mathcal{L}\,u(s) = \int_0^\infty su(t)e^{-st}\,dt = /\ y = st\ / = \int_0^\infty u\left(\frac{y}{s}\right)e^{-y}\,dy.$$

Since $|u(y/s)|e^{-y} \leq Ce^{-y}$ for some constant C (remember that u is bounded), it is clear that

$$\lim_{s\to 0^+}\int_0^\infty u\left(\frac{y}{s}\right)e^{-y}\,dy = \int_0^\infty \lim_{s\to 0^+} u\left(\frac{y}{s}\right)e^{-y}\,dy = \int_0^\infty Ae^{-y}\,dy = A\left[-e^{-y}\right]_0^\infty = A.$$

Initial value theorem

Theorem. Suppose that $u\colon [0,\infty[\to \mathbf{C}$ belongs to X_b and that $u(t) \to a$ as $t \to 0^+$. Then $a = \lim_{\mathbf{R} \ni s \to \infty} s\,\mathcal{L}\,u(s)$.

Proof. Since $u \in X_b$, we know that there exists $C > 0$ such that $|u(t)| \leq Ce^{bt}$. Therefore, let $v(t) = e^{-bt}u(t)$. Let $s \geq 0$ (so real). Then $v(0^+) = u(0^+)$ so we might be able to work with v instead. Indeed, the Laplace transform of v exists and

$$s\,\mathcal{L}\,v(s) = \int_0^\infty sv(t)e^{-st}\,dt = /\ y = st\ / = \int_0^\infty v\left(\frac{y}{s}\right)e^{-y}\,dy.$$

Since $|v(y/s)|e^{-y} \leq Ce^{-y}$ for some constant C (this time $v(t) = u(t)e^{-bt}$ is bounded), it is clear that

$$\lim_{s\to\infty}\int_0^\infty v\left(\frac{y}{s}\right)e^{-y}\,dy = \int_0^\infty \lim_{s\to\infty} v\left(\frac{y}{s}\right)e^{-y}\,dy = \int_0^\infty a\,e^{-y}\,dy = a\left[-e^{-y}\right]_0^\infty = a.$$

Note now that

$$s\,\mathcal{L}\,v(s) = s\,\mathcal{L}(e^{-bt}u(t))(s) = s\,\mathcal{L}\,u(s-b) = (s-b)\,\mathcal{L}\,u(s-b) + b\,\mathcal{L}\,u(s-b),$$

so since $\mathcal{L}\,u(s) \to 0$ as $s \to \infty$, we obtain that

$$\lim_{s\to\infty} s\,\mathcal{L}\,v(s) = \lim_{s\to\infty} (s+b)\,\mathcal{L}\,u(s+b) = \lim_{s\to\infty} s\,\mathcal{L}\,u(s),$$

which proves that $\lim_{s\to\infty} s\,\mathcal{L}\,u(s) = a$. □

We assume that the limits exist!

In the previous two theorems, we *assume* that the limits exist for the result to hold. It can be the case that the limit in the Laplace domain exists (and seems reasonable) but that the limit in the time domain does *not* exist. This can obviously lead to erroneous deductions.

5 More Examples

5.1 Convolution Equations

Example

Solve the equations

$$\int_0^t \cos(t-\tau)u(\tau)\,d\tau = f(t), \quad t \geq 0,$$

where

$$\text{(a) } f(t) = t\sin t \qquad \text{(b) } f(t) = 1.$$

Solution. The integral in question is a convolution of $u(t)$ with $\cos t$. Assuming that $u \in X_a$ for some $a > 0$, we take the Laplace transform of both sides in the equality to find that

$$\frac{s}{s^2+1}\,U(s) = \mathcal{L}\,f(s).$$

If $f(t) = t\sin t$, we obtain

$$\mathcal{L}\,f(s) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2},$$

so

$$U(s) = \frac{2}{s^2+1}.$$

Since $\mathcal{L}(2\sin(t))(s) = U(s)$, the uniqueness result proves that $u(t) = 2\sin t$ is the only solution in X_a.

However, if $f(t) = 1$, we find that

$$\frac{s}{s^2+1}\,U(s) = \frac{1}{s} \quad \Leftrightarrow \quad U(s) = \frac{s^2+1}{s^2} = 1 + \frac{1}{s^2}.$$

Now $\mathcal{L}(t) = s^{-2}$, but what would have the Laplace transform 1? It turns out that there's no such *function* (recall that $\mathcal{L}\,u(s) \to 0$ as $|s| \to \infty$ if $u \in X_a$). We can't find a solution in this case. Plugging the equation into some algebra system might give you the answer $u(t) = t + \delta(t)$, whatever that might mean...

5.2 Power Series

Since it is allowed to integrate a power series termwise (if we're inside the radius of convergence), we can sometimes find the Laplace transform for a function by using a power series representation. It is straight forward to prove that

$$\mathcal{L}(t^m) = \frac{m!}{s^{m+1}}, \quad \operatorname{Re} s > 0,$$

by either using direct calculation and partial integration or by writing $t^m H(t)$ and using the time multiplication theorem. So if we integrate termwise, we can take the Laplace transform of t^m and then calculate the series.

Example

Find the Laplace transform of $u(t) = \operatorname{sinc}(t) = \dfrac{\sin t}{t}$, $t > 0$.

Solution. We have

$$\operatorname{sinc}(t) = \frac{1}{t}\sin t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k+1)!}, \quad t \in \mathbf{R}.$$

Hence

$$\mathcal{L}(\operatorname{sinc}(t)) = \sum_{k=0}^{\infty} \frac{(-1)^k \mathcal{L}(t^{2k})(s)}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{s^{2k+1}(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{s^{2k+1}(2k+1)},$$

which is the Maclaurin series for $\arctan\left(\frac{1}{s}\right)$. This does change the radius of convergence for the power series (there's an exponential weight in the Laplace transform that messes things up). Therefore we have just proved that

$$\mathcal{L}(\operatorname{sinc}(t)) = \arctan\left(\frac{1}{s}\right), \quad \operatorname{Re} s > 1$$

where we leave the difficulties of interpreting this for $s \in \mathbf{C}$ to some other course.

5.3 Bessel Functions

A **Bessel function** of order ν solves Bessel's differential equation:

$$x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y(x) = 0.$$

These are usually denoted by $J_\nu(x)$.

Example

Find the Laplace transform of the solution $J_0(x)$ for $x \geq 0$ such that $J_0(0) = 1$.

Solution. Letting $\nu = 0$ and assuming that $x > 0$, we find that

$$xy''(x) + y'(x) + xy(x) = 0.$$

Assuming that $y, y' \in X_a$ for some $a > 0$, we can take the Laplace transform:

$$\begin{aligned}
&-\frac{d}{ds}\mathcal{L}(y'')(s) + sY(s) - y(0) - \frac{d}{ds}Y(s) = 0 \\
&\quad\Leftrightarrow\quad -\frac{d}{ds}\left(s^2Y(s) - sy(0) - y'(0)\right) + sY(s) - y(0) - Y'(s) = 0 \\
&\quad\Leftrightarrow\quad -2sY(s) - s^2Y'(s) + y(0) + sY(s) - y(0) - Y'(s) = 0 \\
&\quad\Leftrightarrow\quad 0 = sY(s) + (s^2+1)Y'(s) \quad\Leftrightarrow\quad Y'(s) + \frac{s}{s^2+1}Y(s) = 0.
\end{aligned}$$

Now assume for a moment that s is real. Then we can multiply with the integrating factor

$$\exp\left(\frac{1}{2}\ln(s^2+1)\right) = \left(1+s^2\right)^{1/2}$$

so that

$$Y'(s) + \frac{s}{s^2+1}Y(s) = 0 \quad\Leftrightarrow\quad \frac{d}{ds}\left(Y(s)\left(1+s^2\right)^{1/2}\right) = 0 \quad\Leftrightarrow\quad Y(s) = \frac{C}{(1+s^2)^{1/2}}.$$

To find the value of C, consider the limit theorem from above (assuming that y is continuous):

$$1 = y(0) = \lim_{\mathbf{R}\ni s\to\infty} sY(s) = \lim_{\mathbf{R}\ni s\to\infty} C\frac{s}{(1+s^2)^{1/2}} = C.$$

Hence

$$\mathcal{L}\,J_0(s) = \frac{1}{(1+s^2)^{1/2}},$$

again leaving it to a different course how to define this for $s \in \mathbf{C}$.

5.4 Linear Systems of Differential Equations

Suppose that we want to solve, for $t > 0$,

$$\begin{cases} x_1'(t) = 4x_1(t) - 2x_2(t) + e^t, \\ x_2'(t) = 3x_1(t) - 3x_2(t) + e^t, \end{cases}$$

where $x_1(0) = 2/3$ and $x_2(0) = -2$. Taking the Laplace transform, we obtain that

$$\begin{cases} sX_1(s) - \dfrac{2}{3} = 4X_1(s) - 2X_2(s) + \dfrac{1}{s-1}, \\ sX_2(s) + 2 = 3X_1(s) - 3X_2(s) + \dfrac{1}{s-1}, \end{cases} \quad\Leftrightarrow\quad \begin{cases} (s-4)X_1(s) + 2X_2(s) = \dfrac{2}{3} + \dfrac{1}{s-1}, \\ -3X_1(s) + (s+3)X_2(s) = -2 + \dfrac{1}{s-1}. \end{cases}$$

Let

$$A(s) = \begin{pmatrix} s-4 & 2 \\ -3 & s+3 \end{pmatrix}, \qquad X(s) = \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} \quad\text{and}\quad b(s) = \begin{pmatrix} 2/3 + 1/(s-1) \\ -2 + 1/(s-1) \end{pmatrix}$$

so $A(s)X(s) = b(s)$. Assuming that $\det A \neq 0$, we have

$$\begin{aligned} X(s) = A^{-1}b(s) &= \frac{1}{(s-4)(s+3)+6} \begin{pmatrix} s+3 & -2 \\ 3 & s-4 \end{pmatrix} \begin{pmatrix} 2/3 + 1/(s-1) \\ -2 + 1/(s-1) \end{pmatrix} \\ &= \frac{1}{(s-3)(s+2)} \begin{pmatrix} \dfrac{2s^2 + 19s - 15}{3(s+2)} \\ -2s + 11 \end{pmatrix}. \end{aligned}$$

To find something that transforms to this vector, we decompose into partial fractions

$$X_1(s) = \frac{-1/3}{s-1} + \frac{-1}{s+2} + \frac{2}{s-3} \quad \text{and} \quad X_2(s) = \frac{-3}{s+2} + \frac{1}{s-3}.$$

Noting that

$$\mathcal{L}\left(-\frac{1}{3}e^t - e^{-2t} + 2e^{3t}\right) = X_1(s) \quad \text{and} \quad \mathcal{L}\left(-3e^{-2t} + 2e^{3t}\right) = X_2(s),$$

we claim that the unique solution (in X_a) to the problem is given by

$$\begin{cases} x_1(t) = -\frac{1}{3}e^t - e^{-2t} + 2e^{3t}, \\ x_2(t) = -3e^{-2t} + 2e^{3t}. \end{cases}$$

Plugging this into the differential equation to verify that it is a solution might be worth the effort.