Lecture 11: The Unilateral Z-transform

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"Go ahead. Won't show on this shirt"
—Ben Richards

1 Complex Power Series

Let $u_0, u_1, u_2, \ldots$ be a sequence of complex numbers. We will use the notation $u[k] = u_k$. The square brackets indicate that the *function* $u\colon \mathbf{N} \to \mathbf{C}$ is defined on the natural numbers $\mathbf{N}$ (which includes zero in this setting). The **power series** corresponding to this sequence is defined by

$$\sum_{k=0}^{\infty} u[k]z^k = \lim_{n\to\infty} \sum_{k=0}^{n} u[k]z^k,$$

whenever this limit exists. So it is a sequence of complex numbers and convergence is defined on $\mathbf{C}$ as expected (we've used this implicitly already throughout the course).

Existence

Theorem. A complex power series has a radius of convergence R such that $\sum_{k=0}^{\infty} u[k]z^k$ is absolutely convergent if $|z| < R$ and divergent if $|z| > R$.

The behavior when $|z| = R$ (meaning all points in $\mathbf{C}$ of the form $Re^{i\theta}$) is not known at this point (and we will not delve deeper into this subject in this course).

As usual, we find the region of convergence by Cauchy's root-test (or d'Alembert's ratio test). The root-test states that if

$$\limsup_{k\to\infty} \left|u[k]z^k\right|^{1/k} < 1, \tag{1}$$

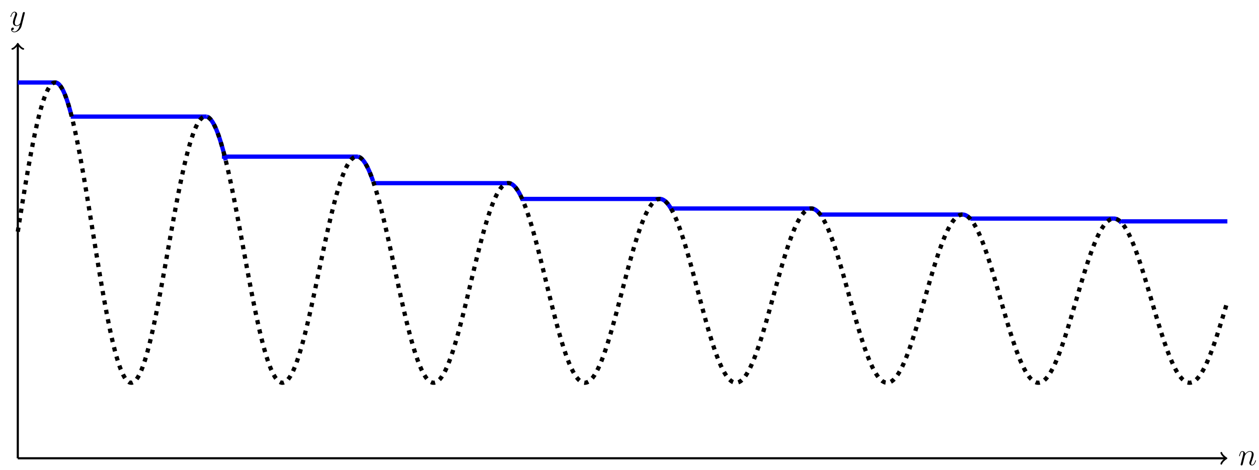
then $\sum_{k=0}^{\infty} u[k]z^k$ is absolutely convergent, and if the limit is > 1, then the series is divergent. In the case when

$$\lim_{k\to\infty} \left|u[k]z^k\right|^{1/k}$$

exists, this limit is equal to the left-hand side of (1) above. We actually find the region of convergence by first calculating the limit and then solving for $|z|$. So why involve the weird *limes superior*? Well, this expression always exist, so that's basically what's needed to ensure that we always have a region of convergence (even if the regular limit doesn't exist like in the case of a sum of geometric series with different quotient). If x_n is a sequence of real numbers, we define $\limsup_{n\to\infty} x_n$ by

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup_{m\geq n} x_m \right).$$

Note that this expression always exist (although it might be $+\infty$) since it is the limit of a decreasing sequence. As an example, consider the two functions below. The dotted curve is the sequence x_n and the blue curve is $\sup_{m\geq n} x_m$, that is, the smallest upper bound to x_m for $m \geq n$.



Uniqueness

Theorem. If $\sum_{k=0}^{\infty} u[k]z^k = \sum_{k=0}^{\infty} v[k]z^k$ for $|z| < R$, where R is some positive constant, then $u[k] = v[k]$ for $k = 0, 1, 2, \ldots$

Basically this means that if two power series converge to the same function in some open disc, then *all* the coefficients must be equal. This is a very useful result.

1.1 Uniform Convergence

We've seen in courses previously that you may differentiate (and integrate) power series termwise. The reason for this is basically that they converge uniformly. To see why the convergence is uniform for $|z| \leq r < R$, note the following. Choose some r_0 such that $r < r_0 < R$. Since $\limsup_{k\to\infty} |u[k]z^k|^{1/k} < 1$, there exists some integer $N > 0$ such that

$$k \geq N \quad \Rightarrow \quad |u[k]z^k|^{1/k} \leq \frac{r}{r_0}.$$

So for $k \geq N$ (and $z \neq 0$), we have

$$|u[k]z^k|^{1/k} \leq \frac{r}{r_0} \quad \Leftrightarrow \quad |u[k]z^k| \leq \left(\frac{r}{r_0}\right)^k.$$

Since $r < r_0 < R$, letting $\rho = r/r_0$ we see that $0 < \rho < 1$ and that

$$|u[k]z^k| \leq \rho^k, \quad k \geq N.$$

Thus

$$\sum_{k=0}^{\infty} |u[k]z^k| \leq \sum_{k=0}^{N-1} |u[k]| r^k + \sum_{k=N}^{\infty} \rho^k < \infty,$$

so by Weierstrass' M-test, the convergence is uniform. So does this mean we can differentiate termwise? Not exactly. However, considering the series of the termwise derivative we see that this also is a power series and that

$$\sum_{k=1}^{\infty} k u[k] z^{k-1} = z^{-1} \sum_{k=0}^{\infty} k u[k] z^k.$$

Note that

$$|ku[k]z^k|^{1/k} = k^{1/k}\, |u[k]z^k|^{1/k}$$

and since $k^{1/k} \to 1$ as $k \to \infty$ it is clear that

$$\limsup_{k\to\infty} |ku[k]z^k|^{1/k} = \limsup_{k\to\infty} |u[k]z^k|^{1/k},$$

so we will obtain the same radius of convergence. Obviously the series of the derivatives also converge uniformly, so yes, we are allowed to differentiate termwise for a power series. Awesome!

2 The Unilateral Z transform

The unilateral Z transform

Definition. For a sequence $u[k]$, $k = 0, 1, 2, \ldots$, we define the **Z transform** of u by

$$\mathcal{Z}(u)(z) = \sum_{k=0}^{\infty} u[k]\, z^{-k},$$

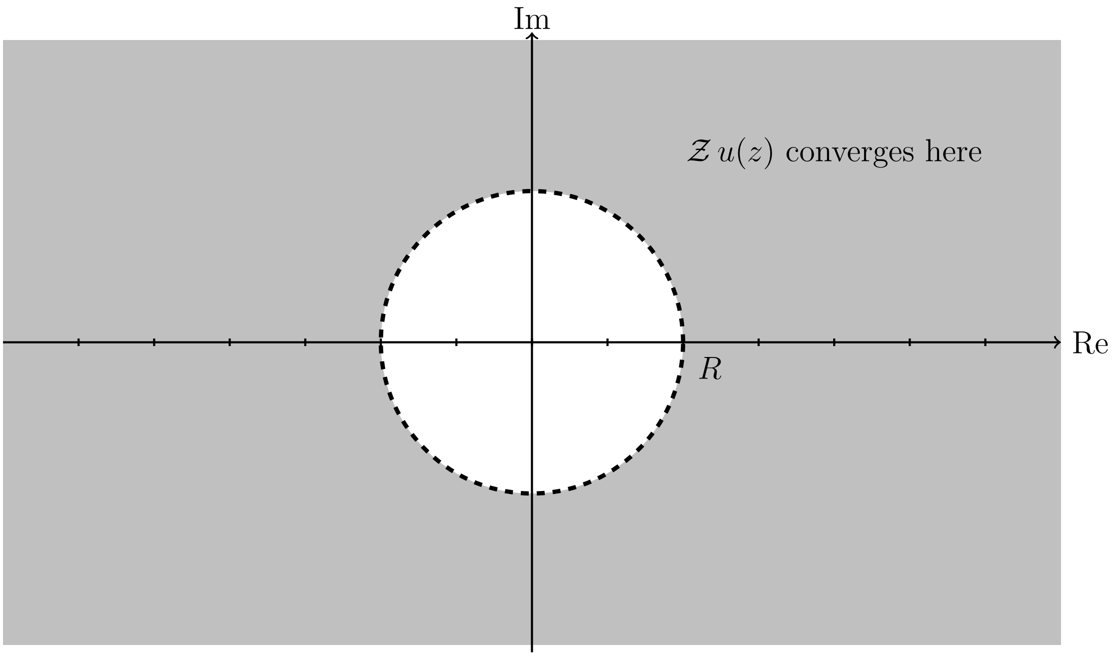
whenever the series is convergent.

We note immediately that this is a power series in the variable z^{-1}. This means that the series has a radius of convergence, but that the series is convergent *outside* a circle with this radius. We have the following result.

Region of convergence

Theorem. For a sequence $u[k]$, $k = 0, 1, 2, \ldots$, the Z transform $\mathcal{Z}\, u(z)$ has a region of convergence defined by the radius of convergence R such that $\mathcal{Z}\, u(z)$ is absolutely (uniformly) convergent for $|z| > R$ and divergent for $|z| < R$. It is possible that $R = 0$.

Proof. This result follows from the existence result for power series by letting $w = z^{-1}$ and considering $\sum_{k=0}^{\infty} u[k]w^k$. □



Example

The geometric series $\sum_{k=0}^{\infty} z^{-k}$ converges if $|z^{-1}| < 1$ and diverges if $|z^{-1}| > 1$. If $|z| > 1$, we have $\sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$.

Impulse function and discrete Heaviside

Definition. We define the discrete impulse function $\delta[n]$ by $\delta[n] = 1$ if $n = 0$ and $\delta[n] = 0$ if $n \neq 0$. The discrete Heaviside function $H[n]$ is defined by $H[n] = 1$ if $n \geq 0$ and $H[n] = 0$ if $n < 0$.

Example

Find the Z transform of $u[n] = \delta[n]$.

Solution. Obviously $\mathcal{Z}(\delta[n])(z) = 1$.

Example

Find the Z transform of $u[k] = 1$, $k = 0, 1, 2, \ldots$

Solution. We find that

$$\mathcal{Z}\, u(z) = \sum_{k=0}^{\infty} \frac{1}{z^k} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}, \quad |z| > 1,$$

since this is a geometric series. Note that this is the Z transform of $H[k]$.

Example

Find the Z transform of $u[k] = k$, $k = 0, 1, 2, \ldots$

Solution. We find that

$$\mathcal{Z}\, u(z) = \sum_{k=0}^{\infty} \frac{k}{z^k} = -z \frac{d}{dz} \sum_{k=0}^{\infty} \frac{1}{z^k} = -z \frac{d}{dz} \frac{z}{z-1} = \frac{z}{(z-1)^2}, \quad |z| > 1,$$

since this is the derivative of a geometric series (remember TATA42?).

Example

Find the Z transform for $u[k] = \dfrac{1}{k!}$.

Solution. We identify the coefficients in the Z transform as those of the Maclaurin series for the exponential function, so

$$\sum_{k=0}^{\infty} \frac{1}{k!} z^{-k} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{z}\right)^k = e^{1/z}.$$

3 Rules for the Z Transform

Linearity

Theorem. Suppose that $\mathcal{Z}\, u(z)$ and $\mathcal{Z}\, v(z)$ exists for $|z| > R$. Then

$$\mathcal{Z}\big(c_1 u[k] + c_2 v[k]\big)(z) = c_1\, \mathcal{Z}\, u(z) + c_2\, \mathcal{Z}\, v(z), \quad |z| > R.$$

Proof. Let $\mathcal{Z}\, u(z)$ and $\mathcal{Z}\, v(z)$ have the radius of convergence R_u and R_v respectively. By defining $R = \max\{R_u,\ R_v\}$, the linearity follows since the summation is linear when all the sums are convergent. □

Geometric multiplier

Theorem. If $a \neq 0$, then $\mathcal{Z}(a^k u[k])(z) = \mathcal{Z}(u[k]) \left(\dfrac{z}{a}\right)$.

Proof. Taking the Z-transform, we find that

$$\mathcal{Z}(a^k u[k])(z) = \sum_{k=0}^{\infty} a^k u[k] z^{-k} = \sum_{k=0}^{\infty} u[k] \left(\frac{z}{a}\right)^{-k} = \mathcal{Z}\, u \left(\frac{z}{a}\right),$$

under the condition that $|z| > |a|R$ where $R > 0$ is the radius of convergence for $\mathcal{Z}\, u(z)$. □

Example

Show that $\mathcal{Z}(a^k) = \dfrac{z}{z-a}$, $|z| > |a|$.

Solution. Recall that $\mathcal{Z}(H) = \dfrac{z}{z-1}$, so by the previous result we obtain that

$$\mathcal{Z}(a^k) = \mathcal{Z}(a^k H(k)) = \frac{z/a}{z/a - 1} = \frac{z}{z-a}, \qquad |z| > |a|.$$

Example

Find the Z-transforms for $\cos k\alpha$ and $\sin k\alpha$.

Solution. Using Euler's equations, we find that

$$\begin{aligned}\mathcal{Z}\left(\frac{e^{ik\alpha}+e^{-ik\alpha}}{2}\right) &= \frac{1}{2}\left(\frac{z}{z-e^{i\alpha}} + \frac{z}{z-e^{-i\alpha}}\right) = \frac{1}{2}\left(\frac{z(z-e^{-i\alpha})+z(z-e^{i\alpha})}{(z-e^{i\alpha})(z-e^{-i\alpha})}\right)\\ &= \frac{1}{2}\left(\frac{2z^2 - z(e^{-i\alpha}+e^{i\alpha})}{z^2 - z(e^{i\alpha}+e^{-i\alpha})+1}\right) = \frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1},\end{aligned}$$

since $\mathcal{Z}(a^k) = z/(z-a)$ for $a \in \mathbf{C}$ $(a \neq 0)$. Analogously,

$$\begin{aligned}\mathcal{Z}\left(\frac{e^{ik\alpha}-e^{-ik\alpha}}{2i}\right) &= \frac{1}{2i}\left(\frac{z}{z-e^{i\alpha}} - \frac{z}{z-e^{-i\alpha}}\right) = \frac{1}{2i}\left(\frac{z(z-e^{-i\alpha})-z(z-e^{i\alpha})}{(z-e^{i\alpha})(z-e^{-i\alpha})}\right)\\ &= \frac{1}{2i}\left(\frac{z(e^{i\alpha}-e^{-i\alpha})}{z^2 - z(e^{i\alpha}+e^{-i\alpha})+1}\right) = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}.\end{aligned}$$

In both cases we have $|z| > 1$.

Conjugation

Theorem. $\mathcal{Z}(\overline{u[k]})(z) = \overline{\mathcal{Z}(u[k])(\overline{z})}$

Proof. Clearly

$$\mathcal{Z}(\overline{u[k]})(z) = \sum_{k=0}^{\infty} \overline{u[k]} z^{-k} = \sum_{k=0}^{\infty} \overline{u[k]\overline{z}^{-k}} = \overline{\sum_{k=0}^{\infty} u[k]\overline{z}^{-k}} = \overline{\mathcal{Z}(u[k])(\overline{z})}$$

for $|z| > R$. □

3.1 Time Shifts

One of the major uses for the Z-transform is its ability to handle delayed signals, meaning expressions of the type $u[k-1]$ etc where we need the value at the previous time step. A special case of this occurs when solving difference equations (we'll see that below).

Time shift

Theorem. For $m > 0$ an integer,

(i) $\mathcal{Z}(u[k+m])(z) = z^m\,\mathcal{Z}(u[k])(z) - \sum_{k=0}^{m-1} u[k]z^{m-k}$,

(ii) $\mathcal{Z}(u[k-m])(z) = z^{-m}\,\mathcal{Z}(u[k])(z) + \sum_{k=-m}^{-1} u[k]z^{-(m+k)}$ (assuming that u is defined for these values) and

(iii) $\mathcal{Z}(u[k-m]H[k-m])(z) = z^{-m}\,\mathcal{Z}(u[k])(z)$.

Proof. We obtain these results by reindexing the series.

(i)

$$\begin{aligned}\mathcal{Z}(u[k+m])(z) &= \sum_{k=0}^{\infty} u[k+m]z^{-k} = \sum_{k=m}^{\infty} u[k]z^{-(k-m)} = z^m \sum_{k=m}^{\infty} u[k]z^{-k} \\ &= z^m\left(\sum_{k=0}^{\infty} u[k]z^{-k} - \sum_{k=0}^{m-1} u[k]z^{-k}\right) = z^m\,\mathcal{Z}\,u(z) - \sum_{k=0}^{m-1} u[k]z^{m-k}.\end{aligned}$$

(ii)

$$\begin{aligned}\mathcal{Z}(u[k-m])(z) &= \sum_{k=0}^{\infty} u[k-m]z^{-k} = \sum_{k=-m}^{\infty} u[k]z^{-k-m} = z^{-m} \sum_{k=-m}^{\infty} u[k]z^{-k} \\ &= z^{-m}\left(\sum_{k=-m}^{-1} u[k]z^{-k} + \sum_{k=0}^{\infty} u[k]z^{-k}\right) = z^{-m}\,\mathcal{Z}\,u(z) + \sum_{k=-m}^{-1} u[k]z^{-m-k}.\end{aligned}$$

(iii)

$$\begin{aligned}\mathcal{Z}(u[k-m]H[k-m])(z) &= \sum_{k=0}^{\infty} u[k-m]H[k-m]z^{-k} = \sum_{k=-m}^{\infty} u[k]H[k]z^{-k-m} \\ &= z^{-m}\sum_{k=0}^{\infty} u[k]z^{-k} = z^{-m}\,\mathcal{Z}\,u(z).\end{aligned}$$

□

3.1.1 Difference Equations

One of the major uses for the (unilateral) Z-transform is solving linear difference equations.

Example

Find a solution to $2u[k+1] - u[k] = 1$, $k = 0, 1, 2, \ldots$, $u[0] = 2$.

Solution. Taking the Z-transform, we find that for $|z| > 1$,

$$2(zU(z) - zu[0]) - U(z) = \frac{z}{z-1} \quad \Leftrightarrow \quad (2z-1)U(z) = 4z + \frac{z}{z-1}$$
$$\Leftrightarrow \quad U(z) = \frac{4z}{2z-1} - \frac{z}{(z-1)(2z-1)}.$$

Using partial fractions, we obtain

$$U(z) = \frac{4z}{2z-1} + \frac{1}{z}\left(\frac{z}{z-1} - \frac{2z}{2z-1}\right).$$

Thus we expect that

$$u[k] = 2 \cdot 2^{-k} + H[k-1]\left(1 - \left(\frac{1}{2}\right)^{k-1}\right) = 2^{1-k} + H[k-1]\left(1 - 2^{1-k}\right).$$

For $k \geq 1$ we have $2^{1-k} + 1 - 2^{1-k} = 1$ and for $k = 0$ we have $u[0] = 2$. Therefore we obtain that $u[k] = 1$ for $k = 1, 2, 3, \ldots$ and $u[0] = 2$. Verify directly!

Example

Find a solution to $u[k+2] + u[k+1] - 2u[k] = 3\delta[k]$, $k = 0, 1, 2, \ldots$, $u[0] = 0$ and $u[1] = 3$.

Solution. Taking the Z-transform, we find that

$$z^2U(z) - z^2u[0] - zu[1] + zU(z) - zu[0] - 2U(z) = 3 \quad \Leftrightarrow \quad (z^2 + z - 2)U(z) = 3 + 3z$$
$$\Leftrightarrow \quad U(z) = \frac{3+3z}{z^2+z-2},$$

at least if $|z| > 2$. Why? Well, $z^2 + z - 2 = (z+2)(z-1)$ so we have poles at -2 and 1 (zeroes of the polynomial in the denominator). Decomposing by partial fractions and reformulating slightly, we find that

$$U(z) = \frac{1}{z+2} + \frac{2}{z-1} = \frac{1}{z}\frac{z}{z+2} + \frac{1}{z}\frac{2z}{z-1}.$$

Similar to the previous example, we obtain that

$$u[k] = (-2)^{k-1}H(k-1) + 2H[k-1]H[k-1] = \begin{cases} (-2)^{k-1} + 2, & \text{for } k \geq 1, \\ 0, & \text{for } k = 0. \end{cases}$$

Verifying, we see that for $k = 0$:

$$u[0+2] + u[0+1] - 2u[0] = (-2)^1 + 2 + 3 - 2 \cdot 0 = 3,$$

for $k = 1$:

$$u[1+2] + u[1+1] - 2u[1] = (-2)^2 + 2 + (-2)^1 + 2 - 2 \cdot 3 = 0,$$

and for $k \geq 2$:

$$u[k+2] + u[k+1] - 2u[k] = (-2)^{k+1} + 2 + (-2)^k + 2 - 2 \cdot \left((-2)^{k-1} + 2\right)$$
$$= (-2)^{k+1} + (-2)^k + (-2)^k = (-2)^k(-2+2) = 0.$$

3.2 Derivatives

Theorem. Suppose that $\mathcal{Z}\,u(z)$ converges for $|z| > R$. Then $\mathcal{Z}(ku[k])(z) = -z\frac{d}{dz}\,\mathcal{Z}(u[k])(z)$ for $|z| > R$.

Proof. We see that

$$\begin{aligned}\mathcal{Z}\,u(z) &= \sum_{k=0}^{\infty} u[k]kz^{-k} = -z\sum_{k=0}^{\infty} u[k]\frac{d}{dz}z^{-k} \\ &= \big/ \text{ uniform convergence } \big/ = -z\frac{d}{dz}\sum_{k=0}^{\infty} u[k]z^{-k} = -z\frac{d}{dz}\,\mathcal{Z}\,u(z), \quad |z| > R,\end{aligned}$$

where some care is needed since this is a complex derivative. □

Example

For $a \in \mathbf{C}$, $a \neq 0$, show that for $|z| > |a|$,

$$\mathcal{Z}(a^k)(z) = \frac{z}{z-a}, \qquad \mathcal{Z}(ka^k)(z) = \frac{az}{(z-a)^2} \quad \text{and} \quad \mathcal{Z}(k^2a^k)(z) = \frac{az^2 + a^2z}{(z-a)^3}.$$

Solution. We find that

$$\mathcal{Z}(a^k)(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|.$$

From this it follows that

$$\mathcal{Z}(ka^k)(z) = -z\frac{d}{dz}\,\mathcal{Z}(a^k)(z) = -z\frac{d}{dz}\frac{z}{z-a} = \frac{az}{(z-a)^2}$$

and that

$$\mathcal{Z}(k^2a^k)(z) = -z\frac{d}{dz}\,\mathcal{Z}(ka^k)(z) = -z\frac{d}{dz}\left(-z\frac{d}{dz}\,\mathcal{Z}(a^k)(z)\right) = -z\frac{d}{dz}\frac{az}{(z-a)^2} = \frac{a^2z + a^2z}{(z-a)^3}.$$

3.3 Binomial Coefficients

Remember that the binomial coefficients were defined by

$$\binom{k}{m} = \frac{k!}{(k-m)!\,m!}, \quad k = 0, 1, 2, \ldots, \; m = 0, 1, 2, \ldots, k.$$

For $k < m$, we let $\binom{k}{m} = 0$ (this might be new).

Theorem. $\mathcal{Z}\left(\left(\begin{array}{c} k \\ m \end{array}\right) a^k\right)(z) = \dfrac{a^m z}{(z-a)^{m+1}}$, $m = 0, 1, 2, \ldots$, $|z| > |a|$.

Proof. First, let's consider the Z-transform of $\left(\begin{array}{c} k \\ m \end{array}\right)$ for some fixed m:

$$\begin{aligned}
\mathcal{Z}\left(\begin{array}{c} k \\ m \end{array}\right)(z) &= \sum_{k=0}^{\infty}\left(\begin{array}{c} k \\ m \end{array}\right) z^{-k} = \frac{1}{m!}\sum_{k=m}^{\infty}\frac{k!}{(k-m)!}z^{-k} \\
&= \frac{1}{m!}\sum_{k=m}^{\infty} k(k-1)(k-2)\cdots(k-m+1)z^{-k} \\
&= \Big/\ w = z^{-1}\ \Big/ = \frac{1}{m!}\sum_{k=m}^{\infty} w^m \frac{d^m}{dw^m} w^k = \frac{w^m}{m!}\frac{d^m}{dw^m}\sum_{k=m}^{\infty} w^k \\
&= \Bigg/\ \frac{d^m}{dw^m}\sum_{k=0}^{m-1} w^k = 0\ \Bigg/ = \frac{w^m}{m!}\frac{d^m}{dw^m}\sum_{k=0}^{\infty} w^k = \frac{w^m}{m!}\frac{d^m}{dw^m}\frac{1}{1-w} \\
&= \frac{w^m}{m!}\frac{m!}{(1-w)^{m+1}} = \frac{w^m}{(1-w)^{m+1}} = \frac{z}{(z-1)^{m+1}}.
\end{aligned}$$

From this it follows that

$$\mathcal{Z}\left(\left(\begin{array}{c} k \\ m \end{array}\right) a^k\right)(z) = \frac{z/a}{(z/a-1)^{m+1}} = \frac{a^m z}{(z-a)^{m+1}}, \quad |z| > |a|. \qquad \square$$

Corollary. $\mathcal{Z}\left(\left(\begin{array}{c} k+n \\ m \end{array}\right) a^k\right)(z) = \dfrac{a^{m-n} z^{n+1}}{(z-a)^{m+1}}$, $m = 1, 2, 3, \ldots$, $n = 0, 1, \ldots, m-1$.

Proof. Since $\left(\begin{array}{c} l \\ m \end{array}\right) = 0$ for $l < m$, it follows that

$$\begin{aligned}
\mathcal{Z}\left(\left(\begin{array}{c} k+n \\ m \end{array}\right) a^k\right)(z) &= a^{-n}\,\mathcal{Z}\left(\left(\begin{array}{c} k+n \\ m \end{array}\right) a^{k+n}\right)(z) = a^{-n} z^n\,\mathcal{Z}\left(\left(\begin{array}{c} k \\ m \end{array}\right) a^k\right)(z) \\
&= a^{-n} z^n \frac{a^m z}{(z-a)^{m+1}} = \frac{a^{m-n} z^{n+1}}{(z-a)^{m+1}}. \qquad \square
\end{aligned}$$

Example

Find a solution to $4u[k+2] - 4u[k+1] + u[k] = 4 \cdot 2^{-k}$, $k = 0, 1, 2, \ldots$, $u[0] = 1$ and $u[1] = 1$.

Solution. Taking the Z-transform of both sides of the equation, we find that for $|z| > 1/2$,

$$
\begin{aligned}
4\left(z^2U(z) - z^2u[0] - zu[1]\right) - 4\left(zU(z) - zu[0]\right) + U(z) &= \frac{4z}{z-\frac{1}{2}} \\
\Leftrightarrow \quad (4z^2 - 4z + 1)U(z) &= \frac{4z}{z-\frac{1}{2}} + 4z^2 \\
\Leftrightarrow \quad U(z) = \frac{z}{\left(z-\frac{1}{2}\right)^3} + \frac{z^2}{\left(z-\frac{1}{2}\right)^2} &= 4\frac{2^{-2}z}{\left(z-\frac{1}{2}\right)^3} + \frac{z^2}{\left(z-\frac{1}{2}\right)^2}.
\end{aligned}
$$

By the previous corollary, we know that

$$
\mathcal{Z}\left(\binom{k}{2} a^k\right)(z) = \frac{a^2 z}{(z-a)^3} \quad \text{and} \quad \mathcal{Z}\left(\binom{k+1}{1} a^k\right)(z) = \frac{z^2}{(z-a)^2},
$$

so by linearity we expect that

$$
u[k] = 4\binom{k}{2}2^{-k} + \binom{k+1}{1}2^{-k} = \left(2k(k-1) + (k+1)\right)2^{-k} = \left(2k^2 - k + 1\right)2^{-k},
$$

for $k = 2, 3, 4, \ldots$, solves the equation.

Verifying, we see that for $k = 0$:

$$
4u[0+2] - 4u[0+1] + u[0] = 7 - 4 + 1 = 4,
$$

for $k = 1$:

$$
u[1+2] + u[1+1] - 2u[1] = 8 - 7 + 1 = 2 = 4 \cdot 2^{-1},
$$

and for $k \geq 2$:

$$
u[k+2] + u[k+1] - 2u[k] = \cdots = 4 \cdot 2^{-k}.
$$