Lecture 12: Inversion, Convolution and Bilateral Transforms

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"I let him go."
—John Matrix

1 Inversion

We first note that the uniqueness of a power series representation (there's only one Maclaurin series) means that we have the following result.

Uniqueness of the Z-transform

Theorem. Suppose that $u[k]$ and $v[k]$ have the same Z-transform, that is, $\mathcal{Z}\,u(z) = \mathcal{Z}\,v(z)$ for all $|z| > R$ for some $R > 0$. Then $u[k] = v[k]$ for $k = 0, 1, 2, \ldots$

For a given Z-transform $U(z)$, we typically find $u[k]$ as we did for the Laplace transform, meaning that we need to rewrite $U(z)$ until we can find the components in a table. The uniqueness then proves that the answer is the only possibility. There is an explicit inversion formula as well:

$$u[k] = \frac{1}{2\pi i} \oint_\gamma z^{k-1} U(z)\, dz,$$

where γ is a closed curve completely inside the region of convergence that loops once around the origin with positive orientation (counter clockwise). Choosing some $r > R$, where R is the radius of convergence of $U(z)$, and letting γ be the closed circle with center at the origin and radius r, i.e., $z = re^{i\theta}$ with $0 \leq \theta \leq 2\pi$, parametrizing the integral above we obtain that

$$u[k] = \frac{1}{2\pi i} \int_0^{2\pi} r^{k-1} e^{i(k-1)\theta}\, U(re^{i\theta}) r i e^{i\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} r^k e^{ik\theta}\, U(re^{i\theta})\, d\theta.$$

Why does this work? Since $U(z)$ is a power series, we are allowed to integrate termwise:

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} r^k e^{ik\theta}\, U(re^{i\theta})\, d\theta &= \frac{1}{2\pi} \int_0^{2\pi} r^k e^{ik\theta} \left(\sum_{m=0}^{\infty} u[m] r^{-m} e^{-im\theta} \right) d\theta \\ &= \frac{1}{2\pi} \sum_{m=0}^{\infty} r^{k-m} u[m] \int_0^{2\pi} e^{i(k-m)\theta}\, d\theta = \frac{1}{2\pi} r^0 u[k] \cdot 2\pi = u[k], \end{aligned}$$

since $\displaystyle\int_0^{2\pi} e^{i(k-m)\theta}\, d\theta = 0$ if $k \neq m$.

This result implies the following theorem. Note that we need a condition for the behavior of U "at infinity."

Theorem. Suppose that $U(z)$ is analytic for $|z| > R$, where $R > 0$ is some constant. If $\lim_{|z|\to\infty} U(z) = A$ for some $A \in \mathbf{C}$, then there exists a unique function $u\colon \mathbf{N} \to \mathbf{C}$ such that $\mathcal{Z}\, u(z) = U(z)$ for $|z| > R$.

Example

Suppose that $U(z) = \log(1 + z^{-1})$, $|z| > 1$. Is there some $u[k]$ such that $U(z) = \mathcal{Z}\, u(z)$?

Solution. So $U(z)$ is a bit problematic if you haven't studied complex analysis, but let's ignore that and just work with this formally. Remember that the Maclaurin series is given by

$$\log(1 + z^{-1}) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} z^{-k}, \quad |z| > 1,$$

so from this it is clear that $u[k] = \dfrac{(-1)^k}{k}$ for $k = 1, 2, 3, \ldots$ By uniqueness, this is the only possibility. Note that $U(z) \to 0$ as $|z| \to \infty$.

2 Discrete Convolution

Definition. For $u, v\colon \mathbf{Z} \to \mathbf{C}$, we define the **convolution** $u * v$ by

$$u * v[n] = \sum_{k=-\infty}^{\infty} u[k]v[n-k],$$

whenever this series exists.

So an obvious question is when this limit exists.

Theorem. If $u, v \in l^1$, then $u * v \in l^1$.

Proof. We first prove that $u * v$ is absolutely integrable:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |u * v[n]| &= \sum_{n=-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty} u[k]v[n-k] \right| \leq \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |u[k]||v[n-k]| \\ &= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |u[k]||v[n-k]| = \sum_{k=-\infty}^{\infty} |u[k]| \sum_{n=-\infty}^{\infty} |v[n-k]|\end{aligned}$$

Note now that

$$\sum_{n=-\infty}^{\infty} |v[n-k]| = /\ m = n - k\ / = \sum_{m=-\infty}^{\infty} |v[m]|,$$

so

$$\sum_{k=-\infty}^{\infty} |u[k]| \sum_{n=-\infty}^{\infty} |v[n-k]| = \left(\sum_{k=-\infty}^{\infty} |u[k]| \right) \left(\sum_{m=-\infty}^{\infty} |v[m]| \right) < \infty.$$

A more compact way of stating this result is that

$$\|u * v\|_{l^1(\mathbf{Z})} \leq \|u\|_{l^1(\mathbf{Z})} \|v\|_{l^1(\mathbf{Z})}.$$

The right-hand side is finite by assumption. □

Did the proof look familiar? It should, go back to lecture 7 and see how we proved the analogous result for the continuous convolution of L^1-functions.

We will soon take a look at the Z-transform of a convolution, and since we're only working with the unilateral Z-transform, we can assume that $u[k] = v[k] = 0$ for $k < 0$. The convolution then reduces to

$$u * v[n] = \sum_{k=0}^{n} u[k]v[n-k], \quad n = 0, 1, 2, \ldots$$

In this case, we can relax the conditions a bit and still obtain convergence.

Unilateral convolution

Theorem. If $u, v \colon \mathbf{N} \to \mathbf{C}$ belong to X_a (meaning that $|u[k]| \leq K a^k$ for some $K > 0$ and $a > 0$), then $u * v \in X_b$ for every $b > a$ and

$$|u * v[k]| \leq C\,(k+1)\,a^k, \quad k \geq 0.$$

Furthermore,

$$\mathcal{Z}(u * v)(z) = \mathcal{Z}\,u(z)\,\mathcal{Z}\,v(z), \quad |z| > a.$$

Proof. By definition, $|u[k]| \leq C_1 a^k$ and $|v[k]| \leq C_2 a^k$, so

$$\begin{aligned} |u * v[n]| &= \left| \sum_{k=0}^{n} u[k]v[n-k] \right| \leq /\text{ monotonicity }/ \leq \sum_{k=0}^{n} |u[k]||v[n-k]| \\ &\leq \sum_{k=0}^{n} C_1 a^k C_2 a^{n-k} = C_1 C_2 (n+1) a^n, \end{aligned}$$

and since $\lim_{n\to\infty} n\delta^{-n} = 0$ for any $\delta > 0$, it follows that $|u * v[n]| \leq C b^n$ for every $b > a$.

So the convolution of u and v is defined and belongs to X_b. Taking the Z-transform, we observe that

$$\begin{aligned} \mathcal{Z}(u * v)(z) &= \sum_{n=0}^{\infty} z^{-n} \sum_{k=0}^{n} u[k]v[n-k] = \sum_{n=0}^{\infty} \sum_{k=0}^{n} z^{-k} u[k] z^{-(n-k)} v[n-k] \\ &= \sum_{k=0}^{\infty} z^{-k} u[k] \sum_{n=k}^{\infty} z^{-(n-k)} v[n-k] = /\ n = m + k\ / = \sum_{k=0}^{\infty} z^{-k} u[k] \sum_{m=0}^{\infty} z^{-m} v[m] \\ &= \mathcal{Z}\,v(z) \sum_{k=0}^{\infty} z^{-k} u[k] = \mathcal{Z}\,v(z)\,\mathcal{Z}\,u(z). \end{aligned}$$

□

Example

Show that $\mathcal{Z}\left(\sum_{k=0}^{n} u[k]\right)(z) = \frac{z}{z-1}\,\mathcal{Z}\,u(z)$.

Solution. Note that

$$\sum_{k=0}^{n} u[k] = \sum_{k=0}^{n} u[k]\cdot 1 = \sum_{k=0}^{n} u[k]\cdot H[n-k] = u * H[n],$$

so

$$\mathcal{Z}\left(\sum_{k=0}^{n} u[k]\right) = \mathcal{Z}(u*H) = (\mathcal{Z}\,u)(z)\cdot\frac{z}{z-1}.$$

Example

Solve the equation $\sum_{k=0}^{n} u[k]3^{-k} = 6^{-n}$, $n = 0, 1, 2, \ldots$

Solution. Note that we can reformulate the equation in terms of a convolution:

$$\sum_{k=0}^{n} u[k]3^{-k} = 6^{-n} \quad\Leftrightarrow\quad \sum_{k=0}^{n} u[k]3^{n-k} = 6^{-n}\cdot 3^{n} = 2^{-n}.$$

Taking the Z-transform, we obtain that

$$\mathcal{Z}\,u(z)\,\mathcal{Z}(3^k H[k])(z) = \frac{z}{z-\frac{1}{2}} \quad\Leftrightarrow\quad U(z)\frac{z}{z-3} = \frac{z}{z-\frac{1}{2}}$$

for $|z| > 3$. Hence

$$U(z) = \frac{z-3}{z-\frac{1}{2}} = 1 - \frac{5/2}{z-\frac{1}{2}} = 1 - \frac{5}{2}\frac{1}{z}\frac{z}{z-\frac{1}{2}}.$$

Therefore it follows that

$$u[k] = \delta[k] - \frac{5}{2}\left(\frac{1}{2}\right)^{k-1} H[k-1],$$

so $u[0] = 1$ and $u[k] = 5\cdot 2^{-k}$ for $k \geq 1$.

3 Limit Results

Initial value theorem

Theorem. If there exists some $R > 0$ such that $\mathcal{Z}\,u(z)$ exists for $|z| > R$, then

$$\lim_{|z|\to\infty} \mathcal{Z}\,u(z) = u[0].$$

Proof. Let $|z| > R$ and put $w = z^{-1}$. Then $|w| < R$ and if $|z| \to \infty$ then $w \to 0$. Since $\mathcal{Z}\, u(z)$ converges uniformly for $|z| > R$, it follows that

$$f(w) = \sum_{k=0}^{\infty} u[k] w^k$$

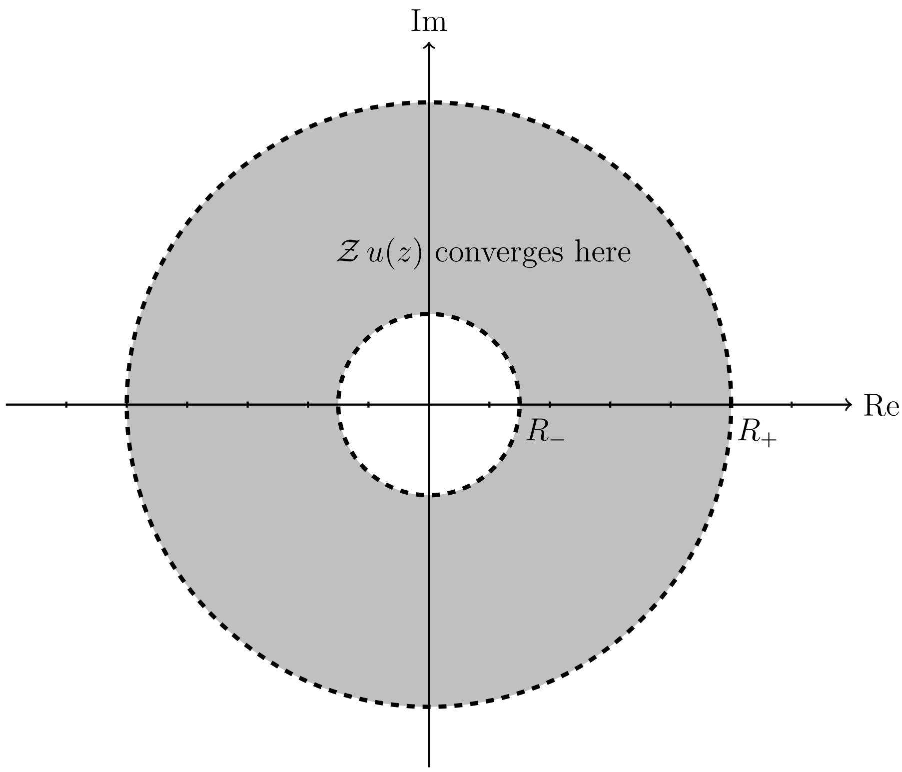
also converges uniformly, so f is a continuous function for $|w| < R$. Hence $\lim_{w\to 0} f(w) = u[0]$ and obviously this limit is the same as $\lim_{|z|\to\infty} \mathcal{Z}\, u(z)$. □

4 The Bilateral Z-transform

We define the **bilateral Z-transform** by

$$\mathcal{Z}\, u(z) = \sum_{k=-\infty}^{\infty} u[k] z^{-k},$$

for those z where the series is absolutely convergent. The theory is similar to the unilateral transform, but some things change. The region of convergence is not only outside of a disc in this case, but also inside another disc (ideally one that's larger..). Hence the region of convergence looks something like this.



Example

Let $u[k] = a^k$ for $k < 0$ and $u[k] = b^k$ for $k \geq 0$. Find $\mathcal{Z}\, u(z)$. When does the transform exist?

Solution. We find that

$$\begin{aligned}\mathcal{Z}\,u(z) &= \sum_{k=-\infty}^{-1} a^k z^{-k} + \sum_{k=0}^{\infty} b^k z^{-k} = \sum_{k=1}^{\infty} a^{-k} z^k + \frac{1}{1-b/z} = \frac{z}{a}\,\frac{1}{1-z/a} + \frac{z}{z-b} \\ &= \frac{z}{a-z} + \frac{z}{z-b},\end{aligned}$$

if $|b| < |z| < |a|$.

Inversion works analogously with the unilateral case, making sure that the integration contour is between the two circles.

5 The Discrete Time Fourier Transform (DTFT)

Recall that $l^1(\mathbf{Z})$ is the space of functions $u\colon \mathbf{Z} \to \mathbf{C}$ such that

$$\sum_{k=-\infty}^{\infty} |u[k]| < \infty,$$

meaning that the sequence is absolutely summable. By considering $z = e^{i\omega}$, the bilateral Z-transform of $u \in l^1(\mathbf{Z})$ takes the form

$$\mathcal{Z}\,u(e^{i\omega}) = \sum_{k=-\infty}^{\infty} u[k] e^{-ik\omega},$$

which is an absolutely convergent series since $u \in l^1(\mathbf{Z})$. We define the **discrete time Fourier transform (DTFT)** as

$$\mathcal{F}\,u(\omega) = \sum_{k=-\infty}^{\infty} u[k] e^{-ik\omega}.$$

In a sense, this is the Fourier transform of a function $u\colon \mathbf{Z} \to \mathbf{C}$. Clearly $\mathcal{F}\,u$ is continuous on $\mathbf{R}$, being the uniformly convergent sum of continuous functions, and it is also 2π-periodic:

$$\mathcal{F}\,u(\omega + 2\pi) = \sum_{k=-\infty}^{\infty} u[k] e^{-ik(\omega+2\pi)} = \sum_{k=-\infty}^{\infty} u[k] e^{-ik\omega} = \mathcal{F}\,u(\omega),$$

since this is true for the exponentials in the sum. Moreover, for $u \in l^1(\mathbf{Z})$, the analogous argument with the inversion of the Z-transform shows that

$$u[k] = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}\,u(\omega) e^{i\omega k}\, d\omega, \quad k \in \mathbf{Z}.$$

Prove this!

5.1 Connection with Fourier Series

Notice that since $\mathcal{F}\,u(\omega)$ is 2π-periodic and continuous, a natural question would be what the Fourier series looks like. Indeed, the Fourier series of $\mathcal{F}\,u$ is connected with u in the following sense. Suppose that $u \in l^1(\mathbf{Z})$ and let $U(\omega) = \mathcal{F}\,u(\omega)$. Then U has the Fourier series

$$U(\omega) \sim \sum_{k=-\infty}^{\infty} u[-k] e^{ik\omega},$$

and if $u \in E$ is continuous and 2π-periodic with

$$u(x) \sim \sum_{k=-\infty}^{\infty} U[k] e^{ik\omega},$$

then $\mathcal{F}\, U(\omega) = u(-\omega)$ assuming that $U \in l^1(\mathbf{Z})$.

6 The Discrete Fourier Transform (DFT)

In the case when a function $u\colon \mathbf{Z} \to \mathbf{C}$ is periodic, meaning that there exists some integer $K > 0$ such that $u[k+K] = u[k]$ for every $k \in \mathbf{Z}$, we can define a variation of the Fourier transform by considering only one period and restricting ourselves to integer values. This variation is usually referred to as the **discrete Fourier transform (DFT)**:

$$\mathcal{F}\, u[n] = \sum_{k=0}^{K-1} u[k] e^{-2\pi i n k/K}, \quad n \in \mathbf{Z}.$$

Clearly $\mathcal{F}\, u$ is periodic: $\mathcal{F}\, u[n+K] = \mathcal{F}\, u[n]$, since

$$\begin{aligned} \mathcal{F}\, u[n+K] &= \sum_{k=0}^{K-1} u[k] e^{-2\pi i (n+K) k/K} = \sum_{k=0}^{K-1} u[k] e^{-2\pi i n k/K} e^{-2\pi i k} = \sum_{k=0}^{K-1} u[k] e^{-2\pi i n k/K} \\ &= \mathcal{F}\, u[n]. \end{aligned}$$

Moreover, since both u and $n \mapsto e^{2\pi i n k/K}$ (for fixed k) are K-periodic, it follows that

$$\mathcal{F}\, u[n] = \sum_{k=M}^{M+K-1} u[k] e^{-2\pi i n k/K}, \quad n \in \mathbf{Z},$$

for any integer M. Note also that the numbers $\omega_k = e^{-2\pi i k/K}$ are the **unit roots**, meaning that for $k = 0, 1, 2, \ldots, K-1$, these numbers are the solutions to the binomial equation $z^K = 1$.

The inversion of the discrete Fourier transform is easily carried out by

$$\mathcal{F}^{-1}\, v[n] = \frac{1}{K} \sum_{k=0}^{K-1} v[k] e^{2\pi i n k/K}, \quad n \in \mathbf{Z}. \tag{1}$$

In the case when we have a function $u\colon \{0, 1, 2, \ldots, K-1\} \to \mathbf{C}$, we proceed like we did when working with Fourier series by considering the periodic extension of u. In this was we can consider the Fourier transform of functions defined on discrete sets. When working with the Fourier transform in applications, this is usually the setting we end up in. Obviously there are a lot of questions as to how the DFT is connected with both the DTFT and the regular Fourier transform on $\mathbf{R}$, but we will not get into these at this point. There are several extremely useful results with regards to sampling of signals that you will see in a course in signal processing.

6.1 Circular Convolution

Recall that the previously studied transforms had the nice property that convolutions usually ended up being the product of the transforms of the factors in the convolution. For the DFT, we basically do this "backwards," meaning that we define an operation $\star$ by

$$(u \star v)[n] = \mathcal{F}^{-1} \left(\mathcal{F}\, u\, \mathcal{F}\, v \right) [n].$$

This operation is usually referred to as **circular convolution**. Why circular? This is due to the periodicity of the involved functions u and v when considered as defined on $\mathbf{Z}$. Indeed,

$$(u \star v)[n] = \sum_{k=0}^{K-1} u[k]\, v[(n-k) \mod K].$$

Here $l \mod K = l$ if $0 \leq l < K$ and $l \mod K = l - mK$ if there exists an integer m such that $0 \leq l - mK < K$.

6.2 Properties

Let $U[n] = \mathcal{F}\, u[n]$ and $V[n] = \mathcal{F}\, v[n]$. Then the following properties hold.

(i) Reversal: $U[K-n] = \mathcal{F}(u[K-k])[n]$.

(ii) Conjugation: $\mathcal{F}(\overline{u})[n] = \overline{U[n]}$.

(iii) Parseval's identity:

$$\frac{1}{K}\sum_{k=0}^{K-1} U[k]\,\overline{V[k]} = \sum_{k=0}^{K-1} u[k]\,\overline{v[k]}.$$

(iv) Multiplication: $\mathcal{F}(uv)[n] = \dfrac{1}{K}(\mathcal{F}\, u \star \mathcal{F}\, v)[n]$ (circular convolution).

(v) $\mathcal{F}^2$: $\mathcal{F}(\mathcal{F}\, u)[n] = Ku[n]$.

6.3 The Fast Fourier Transform (FFT)

The **fast Fourier transform (FFT)** is not yet another transform, but rather a particular way of calculating the DFT. A naive implementation of the DFT shows that for each value n, calculating $\mathcal{F}\, u[n]$ costs performing a sum of K multiplications. Since there are K unique values for n, the cost of finding the complete DFT would be of order $O(K^2)$ (where the constant does not depend on K). This would make finding the Fourier transform rather expensive if K is large.

The revolutionary (it really was) idea of the FFT is to factor the problem into parts, solving these recursively, and thereby obtaining a complexity of order $O(K \log K)$. This is a huge gain. There are many different algorithms for calculating the DFT and those that has a time complexity of order $O(K \log K)$ are referred to as FFT:s. Let's take a look at one way of handling the case when $K = 2^N$ is a power of 2. If the size is not a perfect power of 2, one can use **zero-padding**, meaning that we extend $u[k]$ by zero until we obtain $K = 2^N$ for some N. How would that affect the DFT?

6.3.1 An example when $K = 2^N$

Since $K = 2^N$, we can split $u[k]$ in two parts: when $k = 2l$ is even and when $k = 2l+1$ is odd. Note now that

$$\begin{aligned}
\mathcal{F}\, u[n] &= \sum_{l=0}^{K/2-1} u[2l]e^{-i2\pi n(2l)/K} + \sum_{l=0}^{K/2-1} u[2l+1]e^{-i2\pi n(2l+1)/K} \\
&= \sum_{l=0}^{K/2-1} u[2l]e^{-i2\pi nl/(K/2)} + e^{-i2\pi n/K} \sum_{l=0}^{K/2-1} u[2l+1]e^{-i2\pi nl/(K/2)} \\
&= \mathcal{F}(u[2l])[n] + e^{-i2\pi n/K}\, \mathcal{F}(u[2l+1])[n],
\end{aligned}$$

where the last equality assumes that $0 \le n \le K/2-1$. For $n \ge K/2$, let $n = m + K/2$ for $m = 0, 1, \ldots, K/2-1$. We see that

$$\begin{aligned}
\mathcal{F}\, u[m+K/2] &= \sum_{l=0}^{K/2-1} u[2l]e^{-i2\pi(m+K/2)(2l)/K} + \sum_{l=0}^{K/2-1} u[2l+1]e^{-i2\pi(m+K/2)(2l+1)/K} \\
&= e^{-i2\pi m} \sum_{l=0}^{K/2-1} u[2l]e^{-i2\pi ml/(K/2)} + e^{-i2\pi(m+K/2)/K} \sum_{l=0}^{K/2-1} u[2l+1]e^{-i2\pi ml/(K/2)} \\
&= \mathcal{F}(u[2l])[m] - e^{-i2\pi m/K}\, \mathcal{F}(u[2l+1])[m].
\end{aligned}$$

Hence we have reduced the problem of calculating a DFT of size K to calculating two DFT:s of size $K/2$. This type of recursion will yield a complexity of order $O(K \log K)$.

7 Exercises

1. Prove that
$$u \in l^1(\mathbf{Z}) \quad \Rightarrow \quad u[k] = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{F}\, u(\omega)e^{i\omega k}\, d\omega, \quad k \in \mathbf{Z}.$$
2. Prove that $u[k] = a^{|k|} \in l^1(\mathbf{Z})$ when $|a| < 1$ and find $\mathcal{F}\, u(\omega)$.
3. Prove the inversion formula for the DFT: equation 1.
4. Prove the formulas in Section 6.2.