

# Exercises for the course Graph Theory TATA64

Mostly from Textbooks by Bondy-Murty (1976) and Diestel (2006)

## Notation

$E(G)$  set of edges in  $G$ .

$V(G)$  set of vertices in  $G$ .

$K_n$  complete graph on  $n$  vertices.

$K_{m,n}$  complete bipartite graph on  $m + n$  vertices.

$G^c$  the complement of  $G$ .

$L(G)$  line graph of  $G$ .

$c(G)$  number of components of  $G$  (Note:  $\omega(G)$  in Bondy-Murty).

$o(G)$  number of odd components in  $G$  (i.e. number of components with an odd number of vertices.)

$d_G(v)$  degree of a vertex  $v$  in  $G$ .

$N_G(v)$  set of neighbors in  $G$  of a vertex  $v$ .

$\delta(G)$  minimum degree in  $G$ .

$\Delta(G)$  maximum degree in  $G$ .

$\alpha(G)$  independence number of  $G$ , i.e., the size of the largest independent set in  $G$ .

$\beta(G)$  minimum size of a vertex cover in  $G$ .

$\alpha'(G)$  size of a maximum matching in  $G$ .

$\beta'(G)$  minimum size of an edge cover in  $G$ .

$d_G(u, v)$  distance between  $u$  and  $v$ , i.e., length of a shortest path between  $u$  and  $v$

$\kappa(G)$  connectivity of  $G$ , i.e. the greatest  $k$  such that  $G$  is  $k$ -connected.

$\kappa'(G)$  edge-connectivity of  $G$ , i.e. the greatest  $k$  such that  $G$  is  $k$ -edge-connected. (Note:  $\lambda(G)$  in Diestel)

$\chi(G)$  chromatic number of  $G$ , i.e. minimum  $k$  such that  $G$  has a proper  $k$ -coloring.

$\chi'(G)$  chromatic index (edge-chromatic number) of  $G$ , i.e. minimum  $k$  such that  $G$  has proper  $k$ -edge coloring.

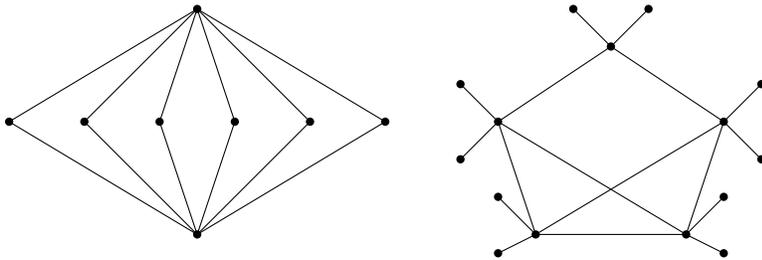
$\omega(G)$  clique number of  $G$ , i.e. the size of a maximum clique in  $G$ .

# 1 Basics. Trees.

- 1.1. Show that if  $G$  is a graph with  $|V(G)| = n$ , then  $|E(G)| \leq \binom{n}{2}$ , with equality if and only if  $G$  is complete.
- 1.2. Show that  $|E(K_{m,n})| = mn$ . Moreover, show that if  $G$  is bipartite, then  $|E(G)| \leq \frac{|V(G)|^2}{4}$ .
- 1.3. The  $k$ -cube  $Q_k$  is the graph whose vertices are the ordered  $k$ -tuples of 0's and 1's, two vertices being joined by an edge if and only if they differ in exactly one coordinate. Show that  $|V(Q_k)| = 2^k$ ,  $|E(G)| = k2^{k-1}$ , and that  $Q_k$  is bipartite.
- 1.4. (a) The *complement*  $G^c$  of a graph  $G$  is the graph with vertex set  $V(G)$ , two vertices being adjacent in  $G^c$  if and only if they are not adjacent in  $G$ . Describe the graphs  $K_n^c$  and  $K_{m,n}^c$ .  
(b)  $G$  is *self-complementary* if  $G \cong G^c$ . Show that if  $G$  is self-complementary, then  $|V(G)| \equiv 0, 1 \pmod{4}$ .
- 1.5. Show that
  - (a) every induced subgraph of a complete graph is complete;
  - (b) every subgraph of a bipartite graph is bipartite.
- 1.6. Show that if a  $k$ -regular bipartite graph with  $k > 0$  has a bipartition  $(X, Y)$ , then  $|X| = |Y|$ .
- 1.7. Show that, in any group of two or more people, there are always two with exactly the same number of friends inside the group.
- 1.8. If a multigraph  $G$  has vertices  $v_1, v_2, \dots, v_n$ , the sequence  $(d(v_1), d(v_2), \dots, d(v_n))$  is called the *degree sequence* of  $G$ . Show that a sequence  $(d_1, d_2, \dots, d_n)$  of non-negative integers is a degree sequence of some multigraph (loops not allowed) if and only if  $\sum_{i=1}^n d_i$  is even and  $d_1 \leq d_2 + \dots + d_n$ .
- 1.9. A sequence  $\mathbf{d} = (d_1, d_2, \dots, d_n)$  is *graphic* if there is a (simple) graph with degree sequence  $\mathbf{d}$ . Show that the sequences  $(7, 6, 5, 4, 3, 3, 2)$  and  $(6, 6, 5, 4, 3, 3, 1)$  are not graphic.
- 1.10. Let  $\mathbf{d} = (d_1, d_2, \dots, d_n)$  be a non-increasing sequence of non-negative integers.
  - (a) Show that  $\mathbf{d}$  is graphic if and only if  $(d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$  is graphic. (Hint: To prove necessity, first show that if  $u_1v_1, u_2v_2 \in E(G)$  and  $u_1v_2, u_2v_1 \notin E(G)$ , then  $G - \{u_1v_1, u_2v_2\} + \{u_1v_2, u_2v_1\}$  has the same degree sequence as  $G$ . Using this, show that if  $\mathbf{d}$  is graphic, then there is a graph  $H$  such that  $V(H) = \{v_1, v_2, \dots, v_n\}$ ,  $d(v_i) = d_i$  for each  $i = 1, \dots, n$ , and  $v_1$  is adjacent to  $v_2, \dots, v_{d_1+1}$ . The graph  $H - v_1$  has degree sequence  $(d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$ .)
  - (b) Using (a), describe an algorithm for constructing a graph with degree sequence  $\mathbf{d}$ , if such a graph exists.
- 1.11. Show that a graph  $G$  contains a spanning bipartite subgraph  $H$  such that  $d_H(v) \geq \frac{1}{2}d_G(v)$  for all  $v \in V(G)$ . (Hint: Show that a bipartite subgraph with the largest possible number of edges has this property.)
- 1.12. Show that if there is a  $(u, v)$ -walk (i.e. a walk beginning at  $u$  and ending at  $v$ ) in  $G$ , then there is also a  $(u, v)$ -path in  $G$ .

- 1.13. (a) Show that if  $G$  is a  $n$ -vertex graph with  $\delta(G) > \lfloor n/2 \rfloor - 1$ , then  $G$  is connected.  
 (b) Find a disconnected  $(\lfloor n/2 \rfloor - 1)$ -regular graph for even  $n$ .
- 1.14. Show that if  $G$  is disconnected, then  $G^c$  is connected.
- 1.15. (a) Show that if  $e \in E(G)$ , then  $c(G) \leq c(G - e) \leq c(G) + 1$ .  
 (b) Let  $v \in V(G)$ . Show that  $G - e$  cannot, in general, be replaced by  $G - v$  in the above inequality.
- 1.16. Show that if  $G$  is a connected graph and every degree in  $G$  is even, then, for any  $v \in V(G)$ ,  $c(G - v) \leq \frac{1}{2}d_G(v)$ .
- 1.17. Show that any two longest paths in a connected graph have a vertex in common.
- 1.18. If vertices  $u$  and  $v$  are connected by a path in  $G$ , the *distance* between  $u$  and  $v$  in  $G$ , denoted by  $d_G(u, v)$ , is the length of a shortest  $(u, v)$ -path in  $G$ ; if there is no path connecting  $u$  and  $v$  we define  $d_G(u, v)$  to be infinite. Show that, for any three vertices  $u, v$  and  $w$ ,  $d(u, v) + d(v, w) \geq d(u, w)$ .
- 1.19. The *diameter* of  $G$  is the maximum distance between two vertices of  $G$ . Show that if  $G$  has diameter greater than three, then  $G^c$  has diameter less than three.
- 1.20. Show that if  $G$  is a graph with diameter two, and  $\Delta(G) = |V(G)| - 2$ , then  $|E(G)| \geq 2|V(G)| - 4$ .
- 1.21. Show that if  $G$  is a connected non-complete graph, then  $G$  has three vertices  $u, v, w$  such that  $uv, vw \in E(G)$  and  $uw \notin E(G)$ .
- 1.22. Show that if an edge  $e$  is in a closed trail of  $G$ , then  $e$  is in a cycle of  $G$ .
- 1.23. Show that if  $G$  is a graph with  $\delta(G) \geq 2$ , then  $G$  contains a cycle of length at least  $\delta(G) + 1$ .
- 1.24. Show that the minor relation  $\preceq$  defines a partial ordering on any set of graphs.
- 1.25. Prove that if a graph  $G$  contains a subdivision of a graph  $H$  as a subgraph, then  $H$  is a minor of  $G$ .
- 1.26. Is there an eulerian graph  $G$  with  $|V(G)|$  even and  $|E(G)|$  odd? Proof or counterexample!
- 1.27. Show that if  $G$  has no vertices of odd degree, then there are edge-disjoint cycles  $C_1, C_2, \dots, C_m$  such that  $E(G) = E(C_1) \cup E(C_2) \cup \dots \cup E(C_m)$ .
- 1.28. Show that if a connected graph  $G$  has  $2k > 0$  vertices of odd degree, then there are  $k$  edge-disjoint trails  $Q_1, Q_2, \dots, Q_k$  in  $G$  such that  $E(G) = E(Q_1) \cup E(Q_2) \cup \dots \cup E(Q_k)$ .
- 1.29. Prove or disprove that every connected graph contains a walk that traverses every edge exactly twice.
- 1.30. Let  $G$  be a (simple) graph.  
 (a) Prove that the number of edges in  $L(G)$  is  $\sum_{v \in V(G)} \binom{d_G(v)}{2}$ .  
 (b) Prove that  $G$  is isomorphic to  $L(G)$  if and only if  $G$  is 2-regular.

- 1.31. Let  $M$  be the incidence matrix and  $A$  the adjacency matrix of a graph  $G$ .
- Show that every column sum of  $M$  is 2.
  - What are the column sums of  $A$ ?
- 1.32. (a) Show that if any two vertices of a graph  $G$  are connected by a unique path, then  $G$  is a tree.
- (b) Prove that the endpoints of a longest path in a nontrivial (i.e. containing at least two vertices) tree both have degree one.
- 1.33. (a) Show that if  $G$  is a tree with  $\Delta(G) \geq k$ , then  $G$  has at least  $k$  vertices of degree one.
- (b) Deduce that every tree with exactly two vertices of degree one is a path.
- 1.34. Let  $G$  be graph with  $|V(G)| - 1$  edges. Show that the following tree statements are equivalent:
- $G$  is connected;
  - $G$  is acyclic;
  - $G$  is a tree.
- 1.35. Show that a sequence  $(d_1, d_2, \dots, d_n)$  of positive integers is a degree sequence of a tree if and only if  $\sum_{i=1}^n d_i = 2(n - 1)$ . (Hint: Use e.g. induction on  $n$ )
- 1.36. Let  $T$  be an arbitrary tree on  $k + 1$  vertices. Show that if  $G$  is a graph with  $\delta(G) \geq k$ , then  $G$  has a subgraph isomorphic to  $T$ .
- 1.37. Show that if  $G$  is a multigraph and has exactly one spanning tree  $T$ , then  $G = T$ .
- 1.38. Let  $F$  be a maximal forest of  $G$  (i.e. a subgraph of  $G$  such that  $F + e$  is not a forest for any  $e \in E(G) \setminus E(F)$ ). Show that
- for every component  $H$  of  $G$ ,  $F \cap H$  is a spanning tree of  $H$ ;
  - $|E(F)| = |V(G)| - c(G)$ .
- 1.39. Find the number of nonisomorphic spanning trees in the following graphs.



- 1.40. Show that
- if every degree in  $G$  is even, then  $G$  has no cut edge;
  - if  $G$  is a  $k$ -regular bipartite graph with  $k \geq 2$ , then  $G$  has no cut edge.
- 1.41. Let  $G$  be a connected graph with at least 3 vertices. Show that
- if  $G$  has a cut edge, then  $G$  has a vertex  $v$  such that  $c(G - v) > c(G)$ ;
  - the converse of (a) is not necessarily true.

- 1.42. Show that a graph that has exactly two vertices which are not cut vertices is a path.
- 1.43. Show that if  $e$  is an edge of  $K_n$ , then the number of spanning trees of  $K_n - e$  is  $(n - 2)n^{n-3}$ .

## 2 Matchings, factors, independent sets and covers

- 2.1. (a) Show that every  $k$ -cube has a perfect matching ( $k \geq 2$ ).  
 (b) Find the number of different perfect matchings in  $K_{2n}$  and  $K_{n,n}$ .
- 2.2. Show that a tree has at most one perfect matching.
- 2.3. Let  $M$  be a matching in a bipartite graph  $G$ . Show that if  $M$  is not maximum, then  $G$  contains an augmenting path with respect to  $M$ .
- 2.4. (a) Prove that every maximal matching in a graph  $G$  has at least  $\alpha'(G)/2$  edges.  
 (b) Let  $G$  be a bipartite graph. Prove that  $\alpha(G) = |V(G)|/2$  if and only if  $G$  has a perfect matching.
- 2.5. For each  $k > 1$ , find an example of a  $k$ -regular multigraph that has no perfect matching. Also, find a cubic (simple) graph without a perfect matching.
- 2.6. Two people play a game on a graph  $G$  by alternately selecting distinct vertices  $v_0, v_1, v_2, \dots$  such that, for  $i > 0$ ,  $v_i$  is adjacent to  $v_{i-1}$ . The last player able to select a vertex wins. Show that the first player has a winning strategy if and only if  $G$  has no perfect matching.
- 2.7. (a) Show that a bipartite graph  $G$  has a perfect matching if and only if  $|N(S)| \geq |S|$  for all  $S \subseteq V(G)$ .  
 (b) Give an example to show that the above statement does not remain valid if the condition that  $G$  be bipartite is dropped.
- 2.8. For  $k > 0$ , show that  
 (a) every  $k$ -regular bipartite graph is 1-factorable.  
 (b) every  $2k$ -regular graph is 2-factorable, i.e., it is the edge-disjoint union of 2-factors.
- 2.9. Let  $A_1, A_2, \dots, A_m$  be subsets of a set  $S$ . A *system of distinct representatives* for the family  $(A_1, A_2, \dots, A_m)$  is a subset  $\{a_1, a_2, \dots, a_m\}$  of  $S$  such that  $a_i \in A_i$ ,  $1 \leq i \leq m$  and  $a_i \neq a_j$  for  $i \neq j$ . Show that  $(A_1, A_2, \dots, A_m)$  has a system of distinct representatives if and only if  $|\bigcup_{i \in J} A_i| \geq |J|$  for all subsets  $J$  of  $\{1, 2, \dots, m\}$ .
- 2.10. Let  $G$  be a  $k$ -regular graph, with  $|V(G)|$  even, that remains connected when any  $k - 2$  edges are deleted. Prove that  $G$  has a 1-factor.
- 2.11. A graph  $G$  is *factor-critical* if each subgraph  $G - v$  obtained by deleting one vertex has a 1-factor. Prove that  $G$  is factor-critical if and only if  $|V(G)|$  is odd and  $o(G - s) \leq |S|$  for all nonempty  $S \subseteq V(G)$ .
- 2.12. A *permutation matrix*  $P$  is a 0, 1-matrix having exactly one 1 in each row and column. Prove that a square matrix of nonnegative integers can be expressed as the sum of  $k$  permutation matrices if and only if all row sums and column sums equal  $k$ .

- 2.13. (a) Show that  $G$  is bipartite if and only if  $\alpha(H) \geq \frac{1}{2}|V(H)|$  for every subgraph  $H$  of  $G$ .  
 (b) Show that  $G$  is bipartite if and only if  $\alpha(H) = \beta'(H)$  for every subgraph  $H$  of  $G$  such that  $\delta(H) > 0$ .
- 2.14. A graph is  $\alpha$ -critical if  $\alpha(G - e) > \alpha(G)$  for all  $e \in E(G)$ . Show that a connected  $\alpha$ -critical graph has no cut-vertices.
- 2.15. For every graph  $G$ , prove that  $\beta(G) \leq 2\alpha'(G)$ . For each  $k \in \mathbf{N}$ , construct a graph with  $\alpha'(G) = k$  and  $\beta(G) = 2k$ .
- 2.16. Let  $G$  be a bipartite graph with parts  $X$  and  $Y$ , such that  $|N(S)| \geq |S|$  whenever  $\emptyset \neq S \subseteq X$ . Prove that every edge of  $G$  belongs to some matching that saturates  $X$ .

### 3 Connectivity. Menger's theorem

- 3.1. (a) Show that if  $G$  is  $k$ -edge connected, with  $k > 0$ , and if  $E'$  is a set of  $k$  edges of  $G$ , then  $c(G - E') \leq 2$ .  
 (b) For all integers  $k > 0$ , find a  $k$ -connected graph  $G$  and a set  $V'$  of  $k$  vertices of  $G$  such that  $c(G - V') > 2$ .
- 3.2. Show that if a graph  $G$  is  $k$ -edge-connected, then  $|E(G)| \geq k|V(G)|/2$ .
- 3.3. (a) Show that if  $G$  is a graph and  $\delta(G) \geq |V(G)| - 2$ , then  $\kappa(G) = \delta(G)$ .  
 (b) Find a simple graph  $G$  with  $\delta(G) = |V(G)| - 3$  and  $\kappa(G) < \delta(G)$ .
- 3.4. Show that if  $G$  is a graph and  $\delta(G) \geq \lfloor |V(G)|/2 \rfloor$ , then  $\kappa'(G) = \delta(G)$ , and prove that this is best possible by constructing for each  $n \geq 4$  an  $n$ -vertex graph with  $\delta(G) = \lfloor n/2 \rfloor - 1$  and  $\kappa'(G) < \delta(G)$ .
- 3.5. Show that if  $G$  is a cubic graph, then  $\kappa'(G) = \kappa(G)$ .
- 3.6. Give an example to show that if  $P$  is a path from  $u$  to  $v$  in a 2-connected graph  $G$ , then  $G$  does not necessarily contain a path  $Q$  from  $u$  to  $v$  that is internally disjoint from  $P$ .
- 3.7. Show that the block graph of any connected graph is a tree.
- 3.8. Show that if  $G$  has no even cycles, then each block of  $G$  is either  $K_1$  or  $K_2$  or an odd cycle.
- 3.9. Let  $G$  be a  $k$ -connected graph, and let  $S, T$  be disjoint subsets of  $V(G)$  with size at least  $k$ . Prove that  $G$  has  $k$  pairwise disjoint  $S, T$ -paths (i.e. a collection of paths the origins of which all lie in  $S$ , and whose termini all lie in  $T$ ).
- 3.10. Let  $G$  be a connected graph in which for every edge  $e$ , there are cycles  $C_1$  and  $C_2$  containing  $e$  whose only common edge is  $e$ . Prove that  $G$  is 3-edge-connected. Use this to show that the Petersen graph is 3-edge-connected.
- 3.11. Prove that a connected graph is  $k$ -edge-connected if and only if each of its blocks is  $k$ -edge-connected.
- 3.12. Let  $k \geq 2$ . Show that a  $k$ -connected graph with at least  $2k$  vertices has a cycle of length at least  $2k$ .

## 4 Vertex colorings. Planar graphs. Turan's theorem

- 4.1. Show that if  $G$  is a graph where any two odd cycles have a vertex in common, then  $\chi(G) \leq 5$ .
- 4.2. Prove that every graph  $G$  has a vertex ordering relative to which the greedy coloring algorithm uses  $\chi(G)$  colors.
- 4.3. Prove that every  $k$ -chromatic graph has at least  $\binom{k}{2}$  edges.
- 4.4. For every  $n > 1$ , find a bipartite graph on  $2n$  vertices, ordered in such a way that the greedy coloring algorithm uses  $n$  rather than 2 colors.
- 4.5. Show that the only 1-critical graph is  $K_1$ , the only 2-critical graph is  $K_2$ , and the only 3-critical graphs are the odd cycles.
- 4.6. Prove that every triangle-free (i.e. not containing a cycle with 3 vertices)  $n$ -vertex graph has chromatic number at most  $2\sqrt{n}$ . (So every  $k$ -chromatic triangle-free graph has at least  $k^2/4$  edges.)
- 4.7. A graph  $G$  is *vertex-color-critical* if  $\chi(G - v) < \chi(G)$  for all  $v \in V(G)$ .
  - (a) Prove that every color-critical graph is vertex-color-critical.
  - (b) Prove that every 3-chromatic vertex-color-critical graph is color-critical.
- 4.8. Let  $G$  be a claw-free graph (i.e. no induced subgraph of  $G$  is isomorphic to  $K_{1,3}$ ).
  - (a) Prove that the subgraph induced by the union of any two color classes in a proper coloring of  $G$  consists of paths and even cycles.
  - (b) Prove that if  $G$  has a proper coloring using exactly  $k$  colors, then  $G$  has a proper  $k$ -coloring where the color classes differ in size by at most one.
- 4.9. Let  $G_3, G_4, \dots$ , be the graphs obtained from  $G_2 = K_2$  using Mycielski's construction. Show that each  $G_k$  is  $k$ -critical.
- 4.10. Show that  $K_{3,3}$  is nonplanar.
- 4.11. (a) Show that  $K_5 - e$  is planar for any edge  $e$  of  $K_5$ .  
(b) Show that  $K_{3,3} - e$  is planar for any edge  $e$  of  $K_{3,3}$ .
- 4.12. Show that a graph is planar if and only if each of its blocks is planar.
- 4.13. A plane graph is *self-dual* if it is isomorphic to its dual.
  - (a) Show that if  $G$  is self-dual, then  $|E(G)| = 2|V(G)| - 2$ .
  - (b) For each  $n \geq 4$ , find a self-dual plane graph on  $n$  vertices.
- 4.14. Let  $G$  be a plane graph. Show that  $(G^*)^*$  is isomorphic to  $G$  (i.e. the dual of the dual of  $G$  is isomorphic to  $G$ ) if and only if  $G$  is connected.
- 4.15. A *plane triangulation* is a plane graph in which each face has degree three. Show that every plane graph is a spanning subgraph of some planar triangulation (if the graph has at least 3 vertices).

- 4.16. The *girth* of a graph is the length of its shortest cycle.
- (a) Show that if  $G$  is a connected planar (simple) graph with girth  $k \geq 3$ , then  $|E(G)| \leq k \frac{|V(G)|-2}{k-2}$ .
- (b) Using (a), show that the Petersen graph is nonplanar.
- 4.17. (a) Show that if  $G$  is a planar graph with at least 11 vertices, then  $G^c$  is nonplanar.
- (b) Find a planar graph  $G$  with 8 vertices, such that  $G^c$  is also planar.
- 4.18. Show that if  $G$  is a plane triangulation, then  $|E(G)| = 3|V(G)| - 6$ .
- 4.19. Show, using Kuratowski's theorem, that the Petersen graph is non-planar.
- 4.20. What does the planar dual of a plane tree look like?
- 4.21. Wagner proved in 1937 that that the following condition is necessary and sufficient for a graph  $G$  to be planar: neither  $K_5$  nor  $K_{3,3}$  can be obtained from  $G$  by performing deletions and contractions of edges.
- (a) Show that deletion and contraction of edges preserve planarity, and conclude that Wagner's conditions is necessary.
- (b) Use Kuratowski's theorem to prove that Wagner's theorem is sufficient.
- 4.22. Use the four color theorem to prove that every planar graph is the edge-disjoint union of two bipartite graphs.
- 4.23. Derive the four color theorem from Hadwiger's conjecture for the case of graphs with chromatic number at least 5.
- 4.24. Prove that a graph is a complete multipartite graph if and only if it has no 3-vertex induced subgraph with one edge.
- 4.25. (a) Show that if  $G$  is a graph and  $|E(G)| > |V(G)|^2/4$ , then  $G$  contains a triangle.
- (b) Find a graph  $G$  with  $|E(G)| = \lfloor |V(G)|^2/4 \rfloor$  that contains no triangle.
- (c) Show that if  $G$  is a non-bipartite graph and  $|E(G)| > (|V(G)| - 1)^2/4 + 1$ , then  $G$  contains a triangle.
- Hint for (c): Assume that  $G$  contains no triangle, and consider a shortest odd cycle  $C$  in  $G$ . Show that each vertex in  $V(G) \setminus V(C)$  can be joined to at most two vertices of  $C$ , and apply (a) to  $G - V(C)$  to obtain a contradiction.
- 4.26. The Turan graph  $T_{n,r}$  is the complete  $r$ -partite with  $b$  partite sets of size  $a + 1$  and  $r - b$  partite sets of size  $a$ , where  $a = \lfloor n/r \rfloor$  and  $b = n - ra$ .
- (a) Prove that  $|E(T_{n,r})| = (1 - 1/r)n^2/2 - b(r - b)/(2r)$ .
- (b) Show that if  $G$  is a complete  $r$ -partite graph on  $n$  vertices, then  $|E(G)| \leq |E(T_{n,r})|$  with equality if and only if  $G$  is isomorphic to  $T_{n,r}$ .
- 4.27. Prove that every  $n$ -vertex graph with no  $(r + 1)$ -clique has at most  $(1 - 1/r)n^2/2$  edges. (Hint: Use the fact that a sum of squares  $f = a_1^2 + a_2^2 + \dots + a_k^2$ , such that  $a_1 + a_2 + \dots + a_k = a$ , is minimized when  $a_i = a/k$  for all  $i$ .)

- 4.28. Let  $G$  be an  $n$ -vertex graph with  $m$  edges.
- (a) Prove that  $\omega(G) \geq \lceil n^2/(n^2 - 2m) \rceil$ . (Hint: Use the previous exercise.)
  - (b) Prove that  $\alpha(G) \geq \lceil n/(d + 1) \rceil$ , where  $d$  is the average degree of  $G$ . (Hint: use part (a).)

## 5 Edge Colorings. Hamilton cycles.

- 5.1. Show, by finding an appropriate edge coloring, that  $\chi'(K_{m,n}) = \Delta(K_{m,n})$ .
- 5.2. Show that the Petersen graph has chromatic index 4.
- 5.3. (a) Show that if  $G$  is bipartite, then  $G$  is contained in a  $\Delta(G)$ -regular bipartite graph.  
 (b) Using (a) and the fact that every regular bipartite graph has a 1-factor, give an alternative proof of König's edge coloring theorem.
- 5.4. Show that if  $G$  is bipartite with  $\delta(G) > 0$ , then  $G$  has a  $\delta(G)$ -edge coloring (not necessarily proper!) such that all  $\delta(G)$  colors are represented at each vertex.
- 5.5. Show by finding appropriate edge colorings, that  $\chi'(K_{2n-1}) = \chi'(K_{2n}) = 2n - 1$ .
- 5.6. (a) Show that if  $G$  is a non-empty regular graph with  $|V(G)|$  odd, then  $\chi'(G) = \Delta(G) + 1$ .  
 (b) Show that if  $G$  is a regular graph with a cut vertex, then  $\chi'(G) > \Delta(G)$ .
- 5.7. (a) Show that if  $G$  is a (loopless) multigraph, then  $G$  is contained in a  $\Delta$ -regular (loopless) multigraph.  
 (b) Using (a) and Petersen's result that every  $2k$ -regular multigraph has a 2-factor, prove that  $\chi'(G) \leq 3\Delta(G)/2$  for any (loopless) multigraph  $G$  with even maximum degree.
- 5.8. Apply Brooks' theorem (not Vizing's) to an 'appropriate' graph to prove that if  $G$  is a graph with  $\Delta(G) = 3$ , then  $\chi'(G) \leq 4$ .
- 5.9. Show that if either
- (a)  $G$  is not 2-connected, or
  - (b)  $G$  is bipartite with bipartition  $(X, Y)$  where  $|X| \neq |Y|$ , then  $G$  is not hamiltonian.
- 5.10. Prove that if  $G$  has a Hamilton path, then  $o(G - S) \leq |S| + 1$ , for every proper subset  $S$  of  $V$ .
- 5.11. A graph  $G$  is called uniquely  $k$ -edge-colorable if any two proper  $k$ -edge colorings of  $G$  induce the same partition of  $E$ . Show that every uniquely 3-edge-colorable 3-regular graph is hamiltonian.
- 5.12. Let  $G$  be a  $n$ -vertex graph that is not a forest and contains no cycles of length less than 5. Prove that the complement of  $G$  is hamiltonian. (Hint: First use Ore's condition on  $G^c$  to deduce that there are vertices  $x, y$  such that  $xy \in E(G)$  and  $d_G(x) + d_G(y) \geq n - 1$ .)
- 5.13. Let  $G$  be a connected graph with  $\delta(G) = k \geq 2$  and  $|V(G)| > 2k$ .
- (a) Let  $P$  be a maximal path in  $G$  (i.e. not a subgraph of any longer path). Prove that if  $|V(P)| \leq 2k$ , then the induced subgraph  $G[V(P)]$  has a spanning cycle.
  - (b) Use part (a) to prove that  $G$  has a path with at least  $2k + 1$  vertices.
- 5.14. A graph is hypohamiltonian if  $G$  is not hamiltonian but  $G - v$  is hamiltonian for every  $v \in V(G)$ . Show that the Petersen graph is hypohamiltonian.

## 6 Ramsey theory

- 6.1. Determine the Ramsey number  $R(3, 3)$ .
- 6.2. Let  $R_n$  denote the Ramsey number  $R(K_3^{(1)}, K_3^{(2)}, \dots, K_3^{(n)})$ , where each  $K_3^{(i)}$  is a triangle (i.e. this Ramsey number is the value of  $r$  such that  $n$ -edge-coloring  $K_r$  forces a monochromatic triangle).
- (a) Show that  $R_n \leq n(R_{n-1} - 1) + 2$ .
  - (b) Noting that  $R_2 = 6$ , use (a) to show that  $R_n \leq \lfloor n!e \rfloor + 1$ .
  - (c) Deduce that  $R_3 \leq 17$ .
- 6.3. Determine the Ramsey number  $R(K_{1,m}, K_{1,n})$ . (Hint: The answer depends on whether  $m$  and  $n$  are even or odd.)
- 6.4. Let  $G_1, G_2, \dots, G_m$  be graphs. The *generalized Ramsey number*  $R(G_1, G_2, \dots, G_m)$  is the smallest integer  $n$  such that every  $m$ -edge coloring of  $K_n$  contains, for some  $i$ , a subgraph isomorphic to  $G_i$  in color  $i$ . Show that  $R(P_4, P_4) = 5$ ,  $R(P_4, C_4) = 5$ , and  $R(C_4, C_4) = 6$ , where  $P_4$  is a 4-vertex path, and  $C_4$  is a 4-vertex cycle.
- 6.5. Prove that if  $T$  is a tree on  $m$  vertices, and  $m - 1$  divides  $n - 1$ , then  $R(T, K_{1,n}) = m + n - 1$ .
- 6.6. Prove that  $R(mK_2, mK_2) = 3m - 1$ , where  $mK_2$  is the graph consisting of  $m$  pairwise disjoint copies of  $K_2$ .