

Exercises for the course Graph Theory TATA64

Mostly from Textbooks by Bondy-Murty (1976) and Diestel (2006)

Notation

$E(G)$ set of edges in G .

$V(G)$ set of vertices in G .

K_n complete graph on n vertices.

$K_{m,n}$ complete bipartite graph on $m + n$ vertices.

G^c the complement of G .

$L(G)$ line graph of G .

$c(G)$ number of components of G (Note: $\omega(G)$ in Bondy-Murty).

$o(G)$ number of odd components in G (i.e. number of components with an odd number of vertices.)

$d_G(v)$ degree of a vertex v in G .

$N_G(v)$ set of neighbors in G of a vertex v .

$\delta(G)$ minimum degree in G .

$\Delta(G)$ maximum degree in G .

$\alpha(G)$ independence number of G , i.e., the size of the largest independent set in G .

$\beta(G)$ minimum size of a vertex cover in G .

$\alpha'(G)$ size of a maximum matching in G .

$\beta'(G)$ minimum size of an edge cover in G .

$d_G(u, v)$ distance between u and v , i.e., length of a shortest path between u and v

$\kappa(G)$ connectivity of G , i.e. the greatest k such that G is k -connected.

$\kappa'(G)$ edge-connectivity of G , i.e. the greatest k such that G is k -edge-connected. (Note: $\lambda(G)$ in Diestel)

$\chi(G)$ chromatic number of G , i.e. minimum k such that G has a proper k -coloring.

$\chi'(G)$ chromatic index (edge-chromatic number) of G , i.e. minimum k such that G has proper k -edge coloring.

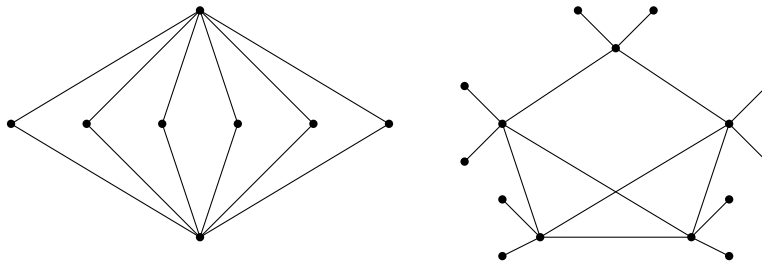
$\omega(G)$ clique number of G , i.e. the size of a maximum clique in G .

1 Basics. Trees.

- 1.1. Show that if G is a graph with $|V(G)| = n$, then $|E(G)| \leq \binom{n}{2}$, with equality if and only if G is complete.
- 1.2. Show that $|E(K_{m,n})| = mn$. Moreover, show that if G is bipartite, then $|E(G)| \leq \frac{|V(G)|^2}{4}$.
- 1.3. The k -cube Q_k is the graph whose vertices are the ordered k -tuples of 0's and 1's, two vertices being joined by an edge if and only if they differ in exactly one coordinate. Show that $|V(Q_k)| = 2^k$, $|E(G)| = k2^{k-1}$, and that Q_k is bipartite.
- 1.4. (a) The *complement* G^c of a graph G is the graph with vertex set $V(G)$, two vertices being adjacent in G^c if and only if they are not adjacent in G . Describe the graphs K_n^c and $K_{m,n}^c$.
 (b) G is *self-complementary* if $G \cong G^c$. Show that if G is self-complementary, then $|V(G)| \equiv 0, 1 \pmod{4}$.
- 1.5. Show that
 - (a) every induced subgraph of a complete graph is complete;
 - (b) every subgraph of a bipartite graph is bipartite.
- 1.6. Show that if a k -regular bipartite graph with $k > 0$ has a bipartition (X, Y) , then $|X| = |Y|$.
- 1.7. Show that, in any group of two or more people, there are always two with exactly the same number of friends inside the group.
- 1.8. If a multigraph G has vertices $v_1, v_2, \dots, v_n$, the sequence $(d(v_1), d(v_2), \dots, d(v_n))$ is called a *degree sequence* of G . Show that a sequence $(d_1, d_2, \dots, d_n)$ of non-negative integers, such that $d_1 \geq d_2 \geq \dots \geq d_n$, is a degree sequence of some multigraph (loops not allowed) if and only if $\sum_{i=1}^n d_i$ is even and $d_1 \leq d_2 + \dots + d_n$.
- 1.9. A sequence $\mathbf{d} = (d_1, d_2, \dots, d_n)$ is *graphic* if there is a (simple) graph with degree sequence $\mathbf{d}$. Show that the sequences $(7, 6, 5, 4, 3, 3, 2)$ and $(6, 6, 5, 4, 3, 3, 1)$ are not graphic.
- 1.10. Let $\mathbf{d} = (d_1, d_2, \dots, d_n)$ be a non-increasing sequence of non-negative integers.
 - (a) Show that $\mathbf{d}$ is graphic if and only if $(d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$ is graphic. (Hint: To prove necessity, first show that if $u_1v_1, u_2v_2 \in E(G)$ and $u_1v_2, u_2v_1 \notin E(G)$, then $G - \{u_1v_1, u_2v_2\} + \{u_1v_2, u_2v_1\}$ has the same degree sequence as G . Using this, show that if $\mathbf{d}$ is graphic, then there is a graph H such that $V(H) = \{v_1, v_2, \dots, v_n\}$, $d(v_i) = d_i$ for each $i = 1, \dots, n$, and v_1 is adjacent to $v_2, \dots, v_{d_1+1}$. The graph $H - v_1$ has degree sequence $(d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$.)
 - (b) Using (a), describe an algorithm for constructing a graph with degree sequence $\mathbf{d}$, if such a graph exists.
- 1.11. Show that a graph G contains a spanning bipartite subgraph H such that $d_H(v) \geq \frac{1}{2}d_G(v)$ for all $v \in V(G)$. (Hint: Show that a bipartite subgraph with the largest possible number of edges has this property.)
- 1.12. Show that if there is a (u, v) -walk (i.e. a walk beginning at u and ending at v) in G , then there is also a (u, v) -path in G .

- 1.13. (a) Show that if G is a n -vertex graph with $\delta(G) > \lfloor n/2 \rfloor - 1$, then G is connected.
 (b) Find a disconnected $(\lfloor n/2 \rfloor - 1)$ -regular graph for even n .
- 1.14. Show that if G is disconnected, then G^c is connected.
- 1.15. (a) Show that if $e \in E(G)$, then $c(G) \leq c(G - e) \leq c(G) + 1$.
 (b) Let $v \in V(G)$. Show that $G - e$ cannot, in general, be replaced by $G - v$ in the above inequality.
- 1.16. Show that if G is a connected graph and every degree in G is even, then, for any $v \in V(G)$, $c(G - v) \leq \frac{1}{2}d_G(v)$.
- 1.17. Show that any two longest paths in a connected graph have a vertex in common.
- 1.18. If vertices u and v are connected by a path in G , the *distance* between u and v in G , denoted by $d_G(u, v)$, is the length of a shortest (u, v) -path in G ; if there is no path connecting u and v we define $d_G(u, v)$ to be infinite. Show that, for any three vertices u, v and w , $d(u, v) + d(v, w) \geq d(u, w)$.
- 1.19. The *diameter* of G is the maximum distance between two vertices of G . Show that if G has diameter greater than three, then G^c has diameter less than three.
- 1.20. Show that if G is a graph with diameter two, and $\Delta(G) = |V(G)| - 2$, then $|E(G)| \geq 2|V(G)| - 4$.
- 1.21. Show that if G is a connected non-complete graph, then G has three vertices u, v, w such that $uv, vw \in E(G)$ and $uw \notin E(G)$.
- 1.22. Show that if an edge e is in a closed trail of G , then e is in a cycle of G .
- 1.23. Show that if G is a graph with $\delta(G) \geq 2$, then G contains a cycle of length at least $\delta(G) + 1$.
- 1.24. Show that the minor relation $\preceq$ defines a partial ordering on any set of graphs.
- 1.25. Prove that if a graph G contains a subdivision of a graph H as a subgraph, then H is a minor of G .
- 1.26. Is there an eulerian graph G with $|V(G)|$ even and $|E(G)|$ odd? Proof or counterexample!
- 1.27. Show that if G has no vertices of odd degree, then there are edge-disjoint cycles $C_1, C_2, \dots, C_m$ such that $E(G) = E(C_1) \cup E(C_2) \cup \dots \cup E(C_m)$.
- 1.28. Show that if a connected graph G has $2k > 0$ vertices of odd degree, then there are k edge-disjoint trails $Q_1, Q_2, \dots, Q_k$ in G such that $E(G) = E(Q_1) \cup E(Q_2) \cup \dots \cup E(Q_k)$.
- 1.29. Prove or disprove that every connected graph contains a walk that traverses every edge exactly twice.
- 1.30. Let G be a (simple) graph.
 (a) Prove that the number of edges in $L(G)$ is $\sum_{v \in V(G)} \binom{d_G(v)}{2}$.
 (b) Prove that G is isomorphic to $L(G)$ if and only if G is 2-regular.

- 1.31. Let M be the incidence matrix and A the adjacency matrix of a graph G .
- Show that every column sum of M is 2.
 - What are the column sums of A ?
- 1.32. (a) Show that if any two vertices of a graph G are connected by a unique path, then G is a tree.
- (b) Prove that the endpoints of a longest path in a nontrivial (i.e. containing at least two vertices) tree both have degree one.
- 1.33. (a) Show that if G is a tree with $\Delta(G) \geq k$, then G has at least k vertices of degree one.
- (b) Deduce that every tree with exactly two vertices of degree one is a path.
- 1.34. Let G be graph with $|V(G)| - 1$ edges. Show that the following tree statements are equivalent:
- G is connected;
 - G is acyclic;
 - G is a tree.
- 1.35. Show that a sequence $(d_1, d_2, \dots, d_n)$ of positive integers is a degree sequence of a tree if and only if $\sum_{i=1}^n d_i = 2(n - 1)$. (Hint: Use e.g. induction on n)
- 1.36. Let T be an arbitrary tree on $k + 1$ vertices. Show that if G is a graph with $\delta(G) \geq k$, then G has a subgraph isomorphic to T .
- 1.37. Show that if G is a multigraph and has exactly one spanning tree T , then $G = T$.
- 1.38. Let F be a maximal forest of G (i.e. a subgraph of G such that $F + e$ is not a forest for any $e \in E(G) \setminus E(F)$). Show that
- for every component H of G , $F \cap H$ is a spanning tree of H ;
 - $|E(F)| = |V(G)| - c(G)$.
- 1.39. Find the number of nonisomorphic spanning trees in the following graphs.



- 1.40. Show that
- if every degree in G is even, then G has no cut edge;
 - if G is a k -regular bipartite graph with $k \geq 2$, then G has no cut edge.

- 1.41. Let G be a connected graph with at least 3 vertices. Show that
- (a) if G has a cut edge, then G has a vertex v such that $c(G - v) > c(G)$;
 - (b) the converse of (a) is not necessarily true.
- 1.42. Show that a graph that has exactly two vertices which are not cut vertices is a path.
- 1.43. Show that if e is an edge of K_n , then the number of spanning trees of $K_n - e$ is $(n - 2)n^{n-3}$.

2 Matchings, factors, independent sets and covers

- 2.1. (a) Show that every k -cube has a perfect matching ($k \geq 2$).
 (b) Find the number of different perfect matchings in K_{2n} and $K_{n,n}$.
- 2.2. Show that a tree has at most one perfect matching.
- 2.3. Let M be a matching in a bipartite graph G . Show that if M is not maximum, then G contains an augmenting path with respect to M .
- 2.4. (a) Prove that every maximal matching in a graph G has at least $\alpha'(G)/2$ edges.
 (b) Let G be a bipartite graph. Prove that $\alpha(G) = |V(G)|/2$ if and only if G has a perfect matching.
- 2.5. For each $k > 1$, find an example of a k -regular multigraph without loops that has no perfect matching. Also, find a cubic (simple) graph without a perfect matching.
- 2.6. Two people play a game on a graph G by alternately selecting distinct vertices $v_0, v_1, v_2, \dots$ such that, for $i > 0$, v_i is adjacent to v_{i-1} . The last player able to select a vertex wins. Show that the first player has a winning strategy if and only if G has no perfect matching.
- 2.7. (a) For $S \subseteq V(G)$, define $N(S) = \cup_{v \in S} N(v)$. Show that a bipartite graph G has a perfect matching if and only if $|N(S)| \geq |S|$ for all $S \subseteq V(G)$.
 (b) Give an example to show that the above statement does not remain valid if the condition that G be bipartite is dropped.
- 2.8. For $k > 0$, show that
- (a) every k -regular bipartite graph is 1-factorable.
 - (b) every $2k$ -regular graph is 2-factorable, i.e., it is the edge-disjoint union of 2-factors.
- 2.9. Let $A_1, A_2, \dots, A_m$ be subsets of a set S . A *system of distinct representatives* for the family $(A_1, A_2, \dots, A_m)$ is a subset $\{a_1, a_2, \dots, a_m\}$ of S such that $a_i \in A_i$, $1 \leq i \leq m$ and $a_i \neq a_j$ for $i \neq j$. Show that $(A_1, A_2, \dots, A_m)$ has a system of distinct representatives if and only if $|\bigcup_{i \in J} A_i| \geq |J|$ for all subsets J of $\{1, 2, \dots, m\}$.
- 2.10. Let G be a k -regular graph, with $|V(G)|$ even, that remains connected when any $k - 2$ edges are deleted. Prove that G has a 1-factor.
- 2.11. A graph G is *factor-critical* if each subgraph $G - v$ obtained by deleting one vertex has a 1-factor. Prove that G is factor-critical if and only if $|V(G)|$ is odd and $o(G - s) \leq |S|$ for all nonempty $S \subseteq V(G)$.

- 2.12. A *permutation matrix* P is a 0, 1-matrix having exactly one 1 in each row and column. Prove that a square matrix of nonnegative integers can be expressed as the sum of k permutation matrices if and only if all row sums and column sums equal k .
- 2.13. (a) Show that G is bipartite if and only if $\alpha(H) \geq \frac{1}{2}|V(H)|$ for every subgraph H of G .
 (b) Show that G is bipartite if and only if $\alpha(H) = \beta'(H)$ for every subgraph H of G such that $\delta(H) > 0$.
- 2.14. A graph is α -critical if $\alpha(G - e) > \alpha(G)$ for all $e \in E(G)$. Show that a connected α -critical graph has no cut-vertices.
- 2.15. For every graph G , prove that $\beta(G) \leq 2\alpha'(G)$. For each $k \in \mathbf{N}$, construct a graph with $\alpha'(G) = k$ and $\beta(G) = 2k$.
- 2.16. Let G be a bipartite graph with parts X and Y , such that $|N(S)| \geq |S|$ whenever $\emptyset \neq S \subseteq X$. Prove that every edge of G belongs to some matching that saturates X .

3 Connectivity. Menger's theorem

- 3.1. (a) Show that if G is k -edge connected, with $k > 0$, and if E' is a set of k edges of G , then $c(G - E') \leq 2$.
 (b) For all integers $k > 0$, find a k -connected graph G and a set V' of k vertices of G such that $c(G - V') > 2$.
- 3.2. Show that if a graph G is k -edge-connected, then $|E(G)| \geq k|V(G)|/2$.
- 3.3. (a) Show that if G is a graph and $\delta(G) \geq |V(G)| - 2$, then $\kappa(G) = \delta(G)$.
 (b) Find a simple graph G with $\delta(G) = |V(G)| - 3$ and $\kappa(G) < \delta(G)$.
- 3.4. Show that if G is a graph and $\delta(G) \geq \lfloor |V(G)|/2 \rfloor$, then $\kappa'(G) = \delta(G)$, and prove that this is best possible by constructing for each $n \geq 4$ an n -vertex graph with $\delta(G) = \lfloor n/2 \rfloor - 1$ and $\kappa'(G) < \delta(G)$.
- 3.5. Show that if G is a cubic graph, then $\kappa'(G) = \kappa(G)$.
- 3.6. Give an example to show that if P is a path from u to v in a 2-connected graph G , then G does not necessarily contain a path Q from u to v that is internally disjoint from P .
- 3.7. Show that the block graph of any connected graph is a tree.
- 3.8. Show that if G has no even cycles, then each block of G is either K_1 or K_2 or an odd cycle.
- 3.9. Let G be a k -connected graph, and let S, T be disjoint subsets of $V(G)$ with size at least k . Prove that G has k pairwise disjoint S, T -paths (i.e. a collection of paths the origins of which all lie in S , and whose termini all lie in T).
- 3.10. Let G be a connected graph in which for every edge e , there are cycles C_1 and C_2 containing e whose only common edge is e . Prove that G is 3-edge-connected. Use this to show that the Petersen graph is 3-edge-connected.
- 3.11. Prove that a connected graph is k -edge-connected if and only if each of its blocks is k -edge-connected.

- 3.12. Let $k \geq 2$. Show that a k -connected graph with at least $2k$ vertices has a cycle of length at least $2k$.

4 Vertex colorings. Planar graphs. Turan's theorem

- 4.1. Show that if G is a graph where any two odd cycles have a vertex in common, then $\chi(G) \leq 5$.
- 4.2. Prove that every graph G has a vertex ordering relative to which the greedy coloring algorithm uses $\chi(G)$ colors.
- 4.3. Prove that every k -chromatic graph has at least $\binom{k}{2}$ edges.
- 4.4. For every $n > 1$, find a bipartite graph on $2n$ vertices, ordered in such a way that the greedy coloring algorithm uses n rather than 2 colors.
- 4.5. Show that the only 2-critical graph is K_2 , and the only 3-critical graphs are the odd cycles.
- 4.6. Prove that every triangle-free (i.e. not containing a cycle with 3 vertices) n -vertex graph has chromatic number at most $2\sqrt{n}$. (So every k -chromatic triangle-free graph has at least $k^2/4$ vertices.) (Hint: Iteratively remove a largest independent set from a triangle-free graph until it has “sufficiently” small maximum degree.)
- 4.7. A graph G is *vertex-color-critical* if $\chi(G - v) < \chi(G)$ for all $v \in V(G)$.
- (a) Prove that every color-critical graph is vertex-color-critical.
- (b) Prove that every 3-chromatic vertex-color-critical graph is color-critical.
- 4.8. Let G be a claw-free graph (i.e. no induced subgraph of G is isomorphic to $K_{1,3}$).
- (a) Prove that every component in the subgraph induced by the union of any two color classes in a proper coloring of G is a path or an even cycle.
- (b) Prove that if G has a proper coloring using exactly k colors, then G has a proper k -coloring where the color classes differ in size by at most one.
- 4.9. Let $G_3, G_4, \dots$, be the graphs obtained from $G_2 = K_2$ using Mycielski's construction. Show that each G_k is k -critical.
- 4.10. Show that $K_{3,3}$ is nonplanar.
- 4.11. (a) Show that $K_5 - e$ is planar for any edge e of K_5 .
- (b) Show that $K_{3,3} - e$ is planar for any edge e of $K_{3,3}$.
- 4.12. Show that a graph is planar if and only if each of its blocks is planar.
- 4.13. A plane graph is *self-dual* if it is isomorphic to its dual.
- (a) Show that if G is self-dual, then $|E(G)| = 2|V(G)| - 2$.
- (b) For each $n \geq 4$, find a self-dual plane graph on n vertices.
- 4.14. Let G be a plane graph. Show that $(G^*)^*$ is isomorphic to G (i.e. the dual of the dual of G is isomorphic to G) if and only if G is connected.

- 4.15. A *plane triangulation* is a plane graph in which each face has degree three. Show that every plane graph is a spanning subgraph of some planar triangulation (if the graph has at least 3 vertices).
- 4.16. The *girth* of a graph is the length of its shortest cycle.
- (a) Show that if G is a connected planar (simple) graph with girth $k \geq 3$, then $|E(G)| \leq k \frac{|V(G)|-2}{k-2}$.
- (b) Using (a), show that the Petersen graph is nonplanar.
- 4.17. (a) Show that if G is a planar graph with at least 11 vertices, then G^c is nonplanar.
- (b) Find a planar graph G with 8 vertices, such that G^c is also planar.
- 4.18. Show that if G is a plane triangulation, then $|E(G)| = 3|V(G)| - 6$.
- 4.19. Show, using Kuratowski's theorem, that the Petersen graph is non-planar.
- 4.20. What does the planar dual of a plane tree look like?
- 4.21. Wagner proved in 1937 that the following condition is necessary and sufficient for a graph G to be planar: neither K_5 nor $K_{3,3}$ can be obtained from G by performing deletions and contractions of edges.
- (a) Show that deletion and contraction of edges preserve planarity, and conclude that Wagner's condition is necessary.
- (b) Use Kuratowski's theorem to prove that Wagner's theorem is sufficient.
- 4.22. Use the four color theorem to prove that every planar graph is the edge-disjoint union of two bipartite graphs.
- 4.23. A graph is *outerplanar* if it has an embedding in the plane where every vertex lies on the boundary of the unbounded face. Use Kuratowski's theorem to prove that G is outerplanar if and only if it has no subgraph that is a subdivision of K_4 or $K_{2,3}$.
- 4.24. Prove that a graph is a complete multipartite graph if and only if it has no 3-vertex induced subgraph with one edge.
- 4.25. (a) Show that if G is a graph and $|E(G)| > |V(G)|^2/4$, then G contains a triangle.
- (b) Find a graph G with $|E(G)| = \lfloor |V(G)|^2/4 \rfloor$ that contains no triangle.
- (c) Show that if G is a non-bipartite graph and $|E(G)| > (|V(G)| - 1)^2/4 + 1$, then G contains a triangle.
- Hint for (c): Assume that G contains no triangle, and consider a shortest odd cycle C in G . Show that each vertex in $V(G) \setminus V(C)$ can be joined to at most two vertices of C , and apply (a) to $G - V(C)$ to obtain a contradiction.
- 4.26. The Turan graph $T_{n,r}$ is the complete r -partite with b partite sets of size $a + 1$ and $r - b$ partite sets of size a , where $a = \lfloor n/r \rfloor$ and $b = n - ra$.
- (a) Prove that $|E(T_{n,r})| = (1 - 1/r)n^2/2 - b(r - b)/(2r)$.
- (b) Show that if G is a complete r -partite graph on n vertices, then $|E(G)| \leq |E(T_{n,r})|$ with equality if and only if G is isomorphic to $T_{n,r}$.

- 4.27. Prove that every n -vertex graph with no $(r+1)$ -clique has at most $(1-1/r)n^2/2$ edges. (Hint: Use the fact that a sum of squares $f = a_1^2 + a_2^2 + \cdots + a_k^2$, such that $a_1 + a_2 + \cdots + a_k = a$, is minimized when $a_i = a/k$ for all i .)
- 4.28. Let G be an n -vertex graph with m edges.
- (a) Prove that $\omega(G) \geq \lceil n^2/(n^2 - 2m) \rceil$. (Hint: Use the previous exercise.)
 - (b) Prove that $\alpha(G) \geq \lceil n/(d+1) \rceil$, where d is the average degree of G . (Hint: use part (a).)
- 4.29. Let G and H be two graphs. Show that $\chi(G \cup H) \leq \chi(G)\chi(H)$. Can you find an example where this upper bound is sharp?

5 Edge Colorings. Hamilton cycles.

- 5.1. Show, by finding an appropriate edge coloring, that $\chi'(K_{m,n}) = \Delta(K_{m,n})$.
- 5.2. Show that the Petersen graph has chromatic index 4.
- 5.3. (a) Show that if G is bipartite, then G is contained in a $\Delta(G)$ -regular bipartite graph.
 (b) Using (a) and the fact that every regular bipartite graph has a 1-factor, give an alternative proof of König's edge coloring theorem.
- 5.4. Show that if G is bipartite with $\delta(G) > 0$, then G has a $\delta(G)$ -edge coloring (not necessarily proper!) such that all $\delta(G)$ colors are represented at each vertex.
- 5.5. Show by finding appropriate edge colorings, that $\chi'(K_{2n-1}) = \chi'(K_{2n}) = 2n - 1$.
- 5.6. (a) Show that if G is a non-empty regular graph with $|V(G)|$ odd, then $\chi'(G) = \Delta(G) + 1$.
 (b) Show that if G is a regular graph with a cut vertex, then $\chi'(G) > \Delta(G)$.
- 5.7. (a) Show that if G is a (loopless) multigraph, then G is contained in a Δ -regular (loopless) multigraph.
 (b) Using (a) and Petersen's result that every $2k$ -regular multigraph has a 2-factor, prove that $\chi'(G) \leq 3\Delta(G)/2$ for any (loopless) multigraph G with even maximum degree.
- 5.8. Apply Brooks' theorem (not Vizing's) to an 'appropriate' graph to prove that if G is a graph with $\Delta(G) = 3$, then $\chi'(G) \leq 4$.
- 5.9. Show that if either
- (a) G is not 2-connected, or
 - (b) G is bipartite with bipartition (X, Y) where $|X| \neq |Y|$,
- then G is not hamiltonian.
- 5.10. Prove that if G has a Hamilton path, then $o(G - S) \leq |S| + 1$, for every proper subset S of V .
- 5.11. A graph G is called uniquely k -edge-colorable if any two proper k -edge colorings of G induce the same partition of E . Show that every uniquely 3-edge-colorable 3-regular graph is hamiltonian.
- 5.12. Let G be a n -vertex graph that is not a forest and contains no cycles of length less than 5. Prove that the complement of G is hamiltonian. (Hint: First use Ore's condition on G^c to deduce that there are vertices x, y such that $xy \in E(G)$ and $d_G(x) + d_G(y) \geq n - 1$.)

- 5.13. Let G be a connected graph with $\delta(G) = k \geq 2$ and $|V(G)| > 2k$.
- (a) Let P be a maximal path in G (i.e. not a subgraph of any longer path). Prove that if $|V(P)| \leq 2k$, then the induced subgraph $G[V(P)]$ has a spanning cycle.
 - (b) Use part (a) to prove that G has a path with at least $2k + 1$ vertices.
- 5.14. A graph is hypohamiltonian if G is not hamiltonian but $G - v$ is hamiltonian for every $v \in V(G)$. Show that the Petersen graph is hypohamiltonian.

6 Ramsey theory

- 6.1. Determine the Ramsey number $R(3, 3)$.
- 6.2. Let R_n denote the Ramsey number $R(K_3^{(1)}, K_3^{(2)}, \dots, K_3^{(n)})$, where each $K_3^{(i)}$ is a triangle (i.e. this Ramsey number is the value of r such that n -edge-coloring K_r forces a monochromatic triangle).
- (a) Show that $R_n \leq n(R_{n-1} - 1) + 2$.
 - (b) Noting that $R_2 = 6$, use (a) to show that $R_n \leq \lfloor n!e \rfloor + 1$.
 - (c) Deduce that $R_3 \leq 17$.
- 6.3. Determine the Ramsey number $R(K_{1,m}, K_{1,n})$. (Hint: The answer depends on whether m and n are even or odd.)
- 6.4. Let $G_1, G_2, \dots, G_m$ be graphs. The *generalized Ramsey number* $R(G_1, G_2, \dots, G_m)$ is the smallest integer n such that every m -edge coloring of K_n contains, for some i , a subgraph isomorphic to G_i in color i . Show that $R(P_4, P_4) = 5$, $R(P_4, C_4) = 5$, and $R(C_4, C_4) = 6$, where P_4 is a 4-vertex path, and C_4 is a 4-vertex cycle.
- 6.5. Prove that if T is a tree on m vertices, and $m - 1$ divides $n - 1$, then $R(T, K_{1,n}) = m + n - 1$.
- 6.6. Prove that $R(mK_2, mK_2) = 3m - 1$, where mK_2 is the graph consisting of m pairwise disjoint copies of K_2 .