Linköpings Universitet
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Kurskod: TATA64
Provkod: TEN1

Examination in Graph Theory

Example Exam kl. 08–12

No aids, no calculators, tables, nor textbooks. Each problem is worth 3 points. To receive full points, a solution needs to be complete. To pass requires 8 points, to get grade 4 requires 12 points, and to get 5 requires 15 points.

1. The k-cube Q_k is the graph whose vertices are the ordered k-tuples of 0's and 1's, two vertices being joined by an edge if and only if they differ in exactly one coordinate. Show that $|V(Q_k)| = 2^k$, $|E(G)| = k2^{k-1}$, and that Q_k is bipartite.

2. Show that a tree T has at least $\Delta(T)$ leaves, where $\Delta(T)$ denotes the maximum degree of T. Use this result to prove that a graph where exactly two vertices are not cut vertices is a path.

3. Let G be a claw-free graph (i.e. no induced subgraph of G is isomorphic to $K_{1,3}$).

   (a) Prove that the subgraph induced by the union of any two color classes in a proper coloring of G consists of paths and even cycles.

   (b) Prove that if G has a proper coloring using exactly k colors, then G has a proper k-coloring where the color classes differ in size by at most one.

4. The Petersen graph is the graph with all subsets of size 2 of $\{1, 2, 3, 4, 5\}$ as vertices, and where two vertices are adjacent if and only if the corresponding subsets are disjoint. Show that the Petersen graph has chromatic index 4.

5. (a) Show that if G is a graph and $\delta(G) \geq |V(G)| - 2$, then $\kappa(G) = \delta(G)$.

   (b) Find a small (simple) graph G with $\delta(G) = |V(G)| - 3$ and $\kappa(G) < \delta(G)$.

6. Let M be a matching in a graph G. Show that if M is not maximum, then G contains an augmenting path with respect to M. Deduce that a matching M in a graph is maximum if and only if the graph contains no augmenting path with respect to M.